Gauge theories on A(dS) space and Killing vectors

Rabin Banerjee *, Bibhas Ranjan Majhi

S.N. Bose National Centre for Basic Sciences, Sector-3, Block-JD, Salt Lake City, Kolkata 700 098, India

Abstract

We provide a general technique for collectively analysing a manifestly covariant formulation of non-abelian gauge theories on both anti-de Sitter as well as de Sitter spaces. This is done by stereographically projecting the corresponding theories, defined on a flat Minkowski space, onto the surface of the A(dS) hyperboloid. The gauge and matter fields in the two descriptions are mapped by conformal Killing vectors and conformal Killing spinors, respectively. A bilinear map connecting the spinors with the vector is established. Different forms of gauge fixing conditions and their equivalence are discussed. The U(1) axial anomaly as well as the non-abelian covariant and consistent chiral anomalies on A(dS) space are obtained. Electric-magnetic duality is demonstrated. The zero curvature limit is shown to yield consistent findings.

Keywords: A(dS) space; Stereographic projection; Killing vectors

1. Introduction

Quantum field theories on anti-de Sitter and de Sitter, collectively denoted as A(dS), space-times have a long history originating from the pioneering paper by Dirac [1]. These space-times are crucial in cosmological studies since they are the only maximally symmetric examples of a curved space-time manifold. Incidentally, the A(dS) space-time is a solution of the negative (positive) cosmological Einstein's equations having the same degree of symmetry as the flat Minkowski space-time solution. Moreover, recently a non-zero

* Corresponding author.
E-mail addresses: rabin@bose.res.in (R. Banerjee), bibhas@bose.res.in (B.R. Majhi).

cosmological constant has been proposed to explain the luminosity observations of the farthest supernovae [2]. The A(dS) metric is therefore expected to play an important role if this proposal is validated. These developments show that the study of field theories on A(dS) space is highly desirable, if not essential.

Most of the available treatments of quantum field theory in curved space-time employ high powered mathematical tools [3]. While exploiting such techniques for the A(dS) case are feasible, it is not particularly practical since it misses the special symmetry properties of this space-time. Examples of such approaches are the dimensional reduction scheme using vierbein language [4,5] or those based on group theoretical notions [6,7]. Yet another method is to use the coordinate independent approach, also called the ambient formalism. For scalar fields this was done in [8,9] which was later extended to include gauge theories [10,11]. While an advantage of this approach is its link (although not a complete one to one mapping) with the corresponding analysis on a flat Minkowski space-time, there is an unpleasant feature which also exists in other approaches [1,6–11]. The point is that whereas the electron wave equation involves the angular momentum operator, the gauge field equation involves both this operator as well as the ordinary momentum operator. Since the A(dS) space is a hyperboloid (pseudosphere) the natural operator entering into the equation of motion should be the relevant angular momentum operator, since translations on the A(dS) space correspond to rotations on the pseudosphere. This is usually corrected by imposing subsidiary conditions to avoid going off the pseudosphere of constant length.

In this paper, we develop a manifestly covariant method of formulating interacting gauge theories on the A(dS) space-time. Some basic features of this method were already discussed by one of us [17] in the context of de Sitter space and its Wick rotated version, the hypersphere which is the n-dimensional sphere immersed in $(n+1)$-dimensional flat space [18,19]. The relevant wave operators always incur the angular momentum operators so that subsidiary conditions necessary in other approaches are avoided. The method is general enough to collectively dicuss both the de Sitter as well as the anti-de Sitter examples. Extention to arbitrary dimensions is straightforward. Our method is applicable for higher rank tensor fields. An exact one to one correspondence with the theories on the flat space has been established. Effectively, the theories on the flat space are projected on to the A(dS) space by a stereographic transformation which is basically a conformal transformation. We show that variables in the gauge sector (like potentials, field strengths, etc.) in the two descriptions are related by rules similar to usual tensor analysis, with the conformal Killing vectors playing the role of the metric. Likewise, quantities in the matter (fermionic) sector are related by the conformal Killing spinors. A bilinear map connects these spinors with the conformal Killing vector. Apart from formulating gauge theories we have also computed the chiral anomalies and exhibited electric-magnetic duality rotations in A(dS) space.

The analysis presented in this paper is basically classical and the extension to quantum field theory will be quite nontrivial. While certain related points are studied in Section 5, fully addressing this issue is beyond the scope of the present paper.

In Section 2, the connection between stereographic projection and conformal Killing vectors is shown including an explicit derivation of the latter using the Cartan-Killing equation. The use of these Killing vectors is also elaborated. The pure Yang–Mills theory on A(dS) space is formulated in details in Section 3. The action is derived. Its equivalence with the standard action defined on an arbitrary curved space is shown by using the explicit form of the induced metric. The gauge symmetry is discussed and its connection with the gauge identity is shown. Different forms of the Lorentz gauge condition and their equivalence are analysed. Finally, implications of subsidiary conditions used in the literature [1,6–11] are mentioned. Section 4 discusses the stereographic projection of the Dirac lagrangian by means of conformal Killing spinors. The bilinear map connecting these spinors with the conformal Killing vector is given. Section 5 provides a detailed calculation of both the U(1) (axial) anomaly as well as the non-abelian (covariant and consistent) chiral anomalies. The counterterm connecting the covariant and consistent anomaly is also computed. Electric-magnetic duality rotations in an abelian theory are discussed in Section 6. A second rank anti-symmetric tensor gauge theory is formulated in Section 7. The zero curvature limit, analysed in Section 8, yields consistent results. The equations of motion on A(dS) space smoothly pass to corresponding equations on the flat Minkowski space. Finally, our conclusions are given in Section 9. An Appendix A discussing the role of boundary conditions has also been included.

2. Stereographic projection and Killing vectors on A(dS) space

Amongst curved space-times, the de Sitter and anti-de Sitter spaces are the only possibilities that have maximal symmetry admitting the highest possible number of Killing vectors. The role of these vectors in suitably defining gauge theories on such spaces is crucial to this analysis. We shall do our discussions for de Sitter and anti-de Sitter spaces collectively.

The A(dS) universe is a pseudosphere in a five-dimensional flat space with Cartesian coordinates $r^a = (r^0, r^1, r^2, r^3, r^4)$ satisfying,

$$r^2 = r_a r^a = \eta_{\mu\nu} r^\mu r^\nu + s(r^4)^2 = sl^2 \tag{1}$$

where $s = -1$ for de Sitter space, $s = +1$ for anti-de Sitter space and l is the A(dS) length parameter. The metric of the de Sitter space $dS(4,1)$ is induced from the pseudoeuclidean metric $\eta = \text{diag}(+1,-1,-1,-1,-1)$. It has the pseudoorthogonal group $SO(4,1)$ as the group of motions. Anti-de Sitter comes from $\eta = \text{diag}(+1,-1,-1,-1,+1)$. It has a pseudoorthogonal group $SO(3,2)$as the group motions.The mostly negative flat Minkowski metric is $\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$ with $\mu,\ \nu = 0,1,2,3$.

It should be mentioned that A(dS) is only locally Weyl (conformally) equivalent to the flat space. Although this is enough for our purposes, we remark that the correspondence is globally more intricate and not discussed here. For instance, Eq. (1) defines a geometry that can be considered to describe a patch covering only a finite time interval in A(dS); it cannot be used to describe field configurations in the full A(dS) geometry.

A useful parametrisation of these spaces is done by exploiting the stereographic projection. The four-dimensional stereographic coordinates (x^μ) are obtained by projecting the A(dS) surface into a target Minkowski space. The relevant equations are [20],

$$r^\mu = \Omega(x) x^\mu; \quad \Omega(x) = \left(1 + s\frac{x^2}{4l^2}\right)^{-1} \tag{2}$$

and,

$$r'^4 = -\Omega(x)\left(1 - s\frac{x^2}{4l^2}\right) \tag{3}$$

where $x^2 = \eta_{\mu\nu}x^\mu x^\nu$ and $r'^4 = \frac{r^4}{l} = s\frac{r_4}{l}$.

The inverse transformation is given by

$$x^\mu = \frac{2}{1 - r'^4}r^\mu \tag{4}$$

In order to define a gauge theory on the A(dS) space analogous stereographic projections for gauge fields have to be obtained. This is done following the method developed by us [18,19] in the example of the hypersphere which was later extended to the de Sitter hyperboloid [17]. The point is that there is a mapping of symmetries on the flat space and the pseudosphere (e.g.translations on the former are rotations on the latter) that is captured by the relevant Killing vectors. Furthermore since stereographic projection is known to be a conformal transformation, one expects that the cherished map among gauge fields would be provided by the conformal Killing vectors. We may write this relation as,

$$\widehat{A}_a = K^\mu_a A_\mu + r_a\phi \tag{5}$$

where the conformal Killing vectors K^μ_a satisfy the transversality condition,

$$r^a K^\mu_a = 0 \tag{6}$$

and an additional scalar field ϕ, which is just the normal component of $\widehat{A}_a$,[1] is introduced,

$$\phi = s\frac{1}{l^2}r^a\widehat{A}_a \tag{7}$$

The five components of $\widehat{A}$ are expressed in terms of the four components of A plus a scalar degree of freedom. To simplify the analysis the scalar field is put to zero. It is straightforward to resurrect it by using the above equations. With the scalar field gone, $\widehat{A}$ is now given by

$$\widehat{A}_a = K^\mu_a A_\mu \tag{8}$$

and satisfies the transversality condition originally used by Dirac [1]

$$r^a\widehat{A}_a = 0 \tag{9}$$

The conformal Killing vectors K^μ_a are now determined. These should satisfy the Cartan-Killing equation which, specialised to a flat four-dimensional manifold, is given by

$$\partial^\nu K^\mu_a + \partial^\mu K^\nu_a = \frac{2}{4}\partial_\lambda K^\lambda_a \eta^{\mu\nu} \tag{10}$$

The most general solution for this equation is given by [21]

$$K^\mu_a = t^\mu_a + \epsilon_a x^\mu + \omega^{\mu\nu}_a x_\nu + \lambda^\mu_a x^2 - 2\lambda^\sigma_a x_\sigma x^\mu \tag{11}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. The various transformations of the conformal group are characterised by the parameters appearing in the above equation; translations by t, dilitations by ϵ, rota-

[1] Hat variables are defined on the A(dS) universe while the normal ones are on the flat space.

tions by ω and inversions (or the special conformal transformations) by λ. Imposing the condition (6) and equating coefficients of terms with distinct powers of x, we find the basic structures of the Killing vectors:

$$K_\nu^\mu = \left(1 + s\frac{x^2}{4l^2}\right)\eta_\nu^\mu - s\frac{x_\nu x^\mu}{2l^2} \tag{12}$$

$$K_4^\mu = sK^{4\mu} = \frac{x^\mu}{l} \tag{13}$$

These Killing vectors establish the link between the A(dS) coordinates and the flat ones by the relation,

$$K_{a\mu} = \left(1 + s\frac{x^2}{4l^2}\right)^2 \frac{\partial r_a}{\partial x^\mu} \tag{14}$$

With the above solution for the Killing vectors, the stereographic projection for the gauge fields (8) is completed leading to, in component form,

$$\widehat{A}_\mu = \left(1 + s\frac{x^2}{4l^2}\right)A_\mu - s\frac{x^\nu x_\mu}{2l^2}A_\nu \tag{15}$$

$$\widehat{A}_4 = \frac{x_\mu}{l}A^\mu \tag{16}$$

The inverse relation is given by

$$\left(1 + s\frac{x^2}{4l^2}\right)A_\mu = \widehat{A}_\mu + s\frac{x_\mu \widehat{A}_4}{2l} \tag{17}$$

Before proceeding to discuss gauge theories some properties of these Killing vectors are summarised. There are two useful relations,

$$K_a^\mu K^{a\nu} = \left(1 + s\frac{x^2}{4l^2}\right)^2 \eta^{\mu\nu} \tag{18}$$

and,

$$K_a^\mu K_{b\mu} = \left(1 + s\frac{x^2}{4l^2}\right)^2 \theta_{ab} = \left(1 + s\frac{x^2}{4l^2}\right)^2 \left(\eta_{ab} - s\frac{r_a r_b}{l^2}\right) \tag{19}$$

These relations are also valid for general D-dimensions. The transverse projector θ_{ab} satisfies,

$$r^a\theta_{ab} = r^b\theta_{ab} = 0; \quad \theta_{ab}\theta^{bc} = \theta_a^c \tag{20}$$

and will be subsequently used in the construction of transverse entities like transverse derivatives or angular momentum operators.

Relation (18) shows that the product of the Killing vectors with repeated 'a' indices yields, up to the conformal factor, the induced metric. The other relation can be interpreted as the transversality condition emanating from (6). For computing derivatives involving Killing vectors, a particularly useful identity is given by

$$K_a^\mu \partial_\mu K^{a\nu} = -\left(1 + s\frac{x^2}{4l^2}\right)s\frac{x^\nu}{l^2} \tag{21}$$

A simple use of (18) yields the inverse of (8) as,

$$A_\mu = \left(1 + s\frac{x^2}{4l^2}\right)^{-2} K_{a\mu}\widehat{A}^a = \frac{\partial r_a}{\partial x^\mu}\widehat{A}^a \tag{22}$$

where the second equality follows from (14). The above relation is an illuminating rephrasing of (17).

Note that the conformal (Jacobian) factor that relates the volume element on the A(dS) space with that in the four-dimensional flat manifold,

$$\mathrm{d}^4x = \mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \left(1 + s\frac{x^2}{4l^2}\right)^4 \mathrm{d}\Omega \tag{23}$$

naturally emerges in (18) and (19). For D-dimensions this generalises to,

$$\mathrm{d}^Dx = \mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}x_2\,\ldots\,\mathrm{d}x_{D-1} = \left(1 + s\frac{x^2}{4l^2}\right)^D \mathrm{d}\Omega \tag{24}$$

The invariant measure is given by

$$\mathrm{d}\Omega = \frac{l}{r_4}\mathrm{d}^4r = \left(\frac{l}{r_4}\right)\mathrm{d}r_0\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3 \tag{25}$$

To observe the use of these Killing vectors, let us analyse the generators of the infinitesimal A(dS) transformations. In terms of the host space Cartesian coordinates r^a, these are written as,

$$L_{ab} = r_a\frac{\partial}{\partial r^b} - r_b\frac{\partial}{\partial r^a} \tag{26}$$

which satisfy the algebra,

$$[L_{ab}, L_{cd}] = \eta_{bc}L_{ad} + \eta_{ad}L_{bc} - \eta_{bd}L_{ac} - \eta_{ac}L_{bd} \tag{27}$$

In terms of the stereographic coordinates the generator is expressed as,

$$L_{ab} = \left(r_aK_b^\mu - r_bK_a^\mu\right)\partial_\mu; \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \tag{28}$$

which can be put in a more illuminating form by contracting with r^a,

$$r^aL_{ab} = sl^2K_b^\mu\partial_\mu \tag{29}$$

clearly showing how rotations on the A(dS) space are connected with the translations on the flat space by the Killing vectors.

An object of related interest is the transverse (or tangent) derivative, expressed in terms of the transverse projector,

$$\nabla^a = \theta^{ab}\frac{\partial}{\partial r^b} \tag{30}$$

This derivative satisfies the properties $r_a\nabla^a = 0$, $\nabla^ar_a = 4$ (or 'D' in D-dimensions) and obeys the following commutation relations,

$$[\nabla^a, \nabla^b] = s\frac{1}{l^2}L^{ab}; \quad [\nabla^a, r^b] = \theta^{ab} \tag{31}$$

Also, the angular momentum operator is directly written in terms of the transverse derivative as,

$$L^{ab} = r^a\nabla^b - r^b\nabla^a; \quad \nabla^a = K^{a\mu}\partial_\mu = s\frac{1}{l^2}r_b L^{ba} \tag{32}$$

where we have used (29).

3. Formulation of Yang–Mills theory on A(dS) space

In this section we discuss the formulation of Yang–Mills theory on the A(dS) space. The theory is obtained by stereographically projecting the usual theory defined on the flat Minkowski space. A comparison with other approaches will be done pointing out the advantages of our formalism.

The pure Yang–Mills theory on the Minkowski space is governed by the standard lagrangian,

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{33}$$

where the field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu] \tag{34}$$

To define the field tensor on the A(dS) space we proceed systematically by looking at the gauge symmetries. If the ordinary potential transforms as,

$$A'_\mu = U^{-1}(A_\mu + \mathrm{i}\partial_\mu)U \tag{35}$$

then the projected potential transforms as,

$$\widehat{A}'_a = K^\mu_a A'_\mu = U^{-1}\left(\widehat{A}_a + s\frac{\mathrm{i}}{l^2}r^b L_{ba}\right)U \tag{36}$$

obtained by using (8) and (29).

The infinitesimal version of these transformations obtained by taking $U = \mathrm{e}^{-\mathrm{i}\lambda}$ is then found to be,

$$\delta A_\mu = D_\mu\lambda = \partial_\mu\lambda - \mathrm{i}[A_\mu, \lambda] \tag{37}$$

for the flat space while for the A(dS) space it is given by

$$\delta\widehat{A}_a = K^\mu_a\delta A_\mu = s\frac{1}{l^2}r^b L_{ba}\lambda - \mathrm{i}[\widehat{A}_a, \lambda] \tag{38}$$

This is put in a more transparent form by introducing, in analogy with the flat space, a 'covariantised angular derivative' [19,24] on the A(dS) space,

$$\widehat{\mathcal{L}}_{ab} = L_{ab} - \mathrm{i}[r_a\widehat{A}_b - r_b\widehat{A}_a,] = -\widehat{\mathcal{L}}_{ba} \tag{39}$$

so that,

$$\delta\widehat{A}_a = s\frac{1}{l^2}r^b\widehat{\mathcal{L}}_{ba}\lambda \tag{40}$$

Note that, this is consistent with $r_a\delta\widehat{A}_a = 0$ which is a consequence of the transversality condition (9).

The covariantised angular derivative satisfies a relation that is the covariantised version of (29),

$$r^a \widehat{\mathcal{L}}_{ab} = sl^2 K^{\mu}_{b} D_{\mu} \tag{41}$$

obtained by using the transversality condition on the gauge fields.

It is feasible to extend the definition (30) of the transverse derivative to the 'covariantised transverse derivative' as,

$$\tilde{\nabla}^a = \nabla^a - \mathrm{i}[\widehat{A}^a,] \tag{42}$$

This is related to the 'covariantised angular momentum' (39) by equations similar to (31) and (32). These are given by their covariantised versions,

$$\tilde{\nabla}^a = K^{a\mu} D_{\mu} = K^{a\mu}(\partial_{\mu} - \mathrm{i}[A_{\mu},]) = \frac{s}{l^2} r_b \widehat{\mathcal{L}}^{ba} \tag{43}$$

and,

$$[\tilde{\nabla}^a, r^b] = \theta^{ab}$$
$$[\tilde{\nabla}^a, \tilde{\nabla}^b] = \frac{s}{l^2} \widehat{\mathcal{L}}^{ab}$$

Also, $\tilde{\nabla}^a$ satisfies the following properties that are exactly identical to ∇^a;

$$\begin{aligned} r_a \tilde{\nabla}^a &= r_a \nabla^a = 0 \\ \tilde{\nabla}^a r_a &= \nabla^a r_a = 4 \end{aligned} \tag{44}$$

The field tensor $\widehat{F}_{abc}$ on the A(dS) space is now defined. It has to be a fully anti-symmetric three index object that transforms covariantly. The covariantised angular derivative is the natural choice for constructing it. We define,

$$\widehat{F}_{abc} = \left(L_{ab} \widehat{A}_c - \mathrm{i} r_a [\widehat{A}_b, \widehat{A}_c] \right) + \text{c.p.} \tag{45}$$

where c.p. stands for the other pair of terms involving cyclic permutations in a, b, c.

To see that $\widehat{F}_{abc}$ transforms covariantly it is convenient to recast this in a form involving the Killing vectors, analogous to the relation (8). Indeed it is mapped to the field tensor on the flat space by the following relation,

$$\widehat{F}_{abc} = \left(r_a K^{\mu}_{b} K^{\nu}_{c} + r_b K^{\mu}_{c} K^{\nu}_{a} + r_c K^{\mu}_{a} K^{\nu}_{b} \right) F_{\mu\nu} \tag{46}$$

so that symmetry properties under exchange of the indices is correctly preserved. To show the equivalence, (8) and (28) are used to simplify (45), yielding,

$$\widehat{F}_{abc} = \left(r_a K^{\mu}_{b} - r_b K^{\mu}_{a} \right) \partial_{\mu} \left(K^{\nu}_{c} A_{\nu} \right) - \mathrm{i} r_a \left[K^{\nu}_{b} A_{\nu}, K^{\mu}_{c} A_{\mu} \right] + \text{c.p.} \tag{47}$$

The derivatives acting on the Killing vectors sum up to zero on account of the identity,

$$\left(r_a K^{\mu}_{b} - r_b K^{\mu}_{a} \right) \partial_{\mu} K^{\nu}_{c} + \text{c.p.} = 0 \tag{48}$$

The derivatives acting on the potentials, together with the other pieces, combine to reproduce (46), thereby completing the proof of the equivalence.It is now trivial to see, using the above relation (46), that $\widehat{F}_{abc}$ transforms covariantly simply because $F_{\mu\nu}$ does. The exact transformation is given by,

$$\delta\widehat{F}_{abc} = -\mathrm{i}[\widehat{F}_{abc}, \lambda] \tag{49}$$

Of course this can also be derived by starting from (45) and using (38).

The inverse relation following from (46), obtained by contracting with r^a and the Killing vectors, is given by

$$F_{\mu_\nu}(x) = s\frac{1}{l^2}\left(1 + s\frac{x^2}{4l^2}\right)^{-4} K_{b\mu}K_{c\nu}(r_a\widehat{F}^{abc}) = s\frac{1}{l^2}\frac{\partial r_b}{\partial x^\mu}\frac{\partial r_c}{\partial x^\nu}(r_a\widehat{F}^{abc}(r)) \tag{50}$$

where use has been made of (14) to get the final result.

Incidentally, the generalisation of (22) and (50) to arbitrary rank tensors is easily done,

$$A_{\mu_1\mu_2\ldots\mu_n}(x) = \frac{\partial r_{a_1}}{\partial x^{\mu_1}}\frac{\partial r_{a_2}}{\partial x^{\mu_2}}\cdots\frac{\partial r_{a_n}}{\partial x^{\mu_n}}\widehat{A}^{a_1a_2\ldots a_n}(r) \tag{51}$$

and

$$F_{\mu_1\mu_2\ldots\mu_n\mu_{n+1}}(x) = s\frac{1}{l^2}\frac{\partial r_{b_1}}{\partial x^{\mu_1}}\frac{\partial r_{b_2}}{\partial x^{\mu_2}}\cdots\frac{\partial r_{b_{n+1}}}{\partial x^{\mu_{n+1}}}(r_a\widehat{F}^{ab_1b_2\ldots b_{n+1}}(r)) \tag{52}$$

where $F(\widehat{F})$ denotes the field strength corresponding to the potential $A(\widehat{A})$.

It is possible to extend the above relations to establish a mapping between the covariant derivatives on the A(dS) local coordinates and the transverse derivatives (30) or the angular momentum operator (32). Let us first recall that the A(dS) space is immersed in a $(4+1)$-dimensional flat space which has the metric $\mathrm{d}s^2 = \eta_{ab}\mathrm{d}r^a\mathrm{d}r^b$ and A(dS) is the subspace $\eta_{ab}r^ar^b = sl^2$ with metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. So, locally both metrics must agree on A(dS); i.e.

$$\eta_{ab}\,\mathrm{d}r^a\,\mathrm{d}r^b = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{53}$$

But,

$$\mathrm{d}r^a = \partial_\mu r^a\,\mathrm{d}x^\mu = \frac{\partial r^a}{\partial x^\mu}\,\mathrm{d}x^\mu = \left(1 + s\frac{x^2}{4l^2}\right)^{-2} K^a_\mu\,\mathrm{d}x^\mu \tag{54}$$

Hence,

$$\left(1 + s\frac{x^2}{4l^2}\right)^{-4}\eta^{ab}K_{a\mu}K_{b\nu} = g_{\mu\nu} \tag{55}$$

Using the identity (18), we obtain the induced metric and its inverse,

$$g_{\mu\nu} = \left(1 + s\frac{x^2}{4l^2}\right)^{-2}\eta_{\mu\nu}; \quad g^{\mu\nu} = \left(1 + s\frac{x^2}{4l^2}\right)^{2}\eta^{\mu\nu}. \tag{56}$$

Incidentally, the induced metric $g_{\mu\nu}$ is related to the transverse projector θ_{ab} (see (19)) by

$$g_{\mu\nu} = \frac{\partial r^a}{\partial x^\mu}\frac{\partial r^b}{\partial x^\nu}\theta_{ab} \tag{57}$$

which is similar to (50). This follows from (55), the identification (14) and the transversality condition (6).

The Affine connection for the vector field,

$$\Gamma^\nu_{\mu\sigma} = \frac{1}{2}g^{\nu\rho}(g_{\rho\mu,\sigma} + g_{\rho\sigma,\mu} - g_{\mu\sigma,\rho}) \tag{58}$$

is then found to be,

$$\Gamma^{\nu}_{\mu\sigma} = s\frac{1}{2l^2}\left(1+s\frac{x^2}{4l^2}\right)^{-1}\left(x^{\nu}\eta_{\mu\sigma} - x_{\mu}\eta^{\nu}_{\sigma} - x_{\sigma}\eta^{\nu}_{\mu}\right) \tag{59}$$

The covariant derivative acting on a scalar and a covariant vector yields,

$$\nabla_{\mu}\phi = \partial_{\mu}\phi \tag{60}$$

and

$$\nabla_{\mu}A_{\nu} = \partial_{\mu}A_{\nu} - \Gamma^{\sigma}_{\mu\nu}A_{\sigma} = \partial_{\mu}A_{\nu} - s\frac{1}{2l^2}\left(1+s\frac{x^2}{4l^2}\right)^{-1}(x_{\sigma}A^{\sigma}\eta_{\mu\nu} - x_{\mu}A_{\nu} - x_{\nu}A_{\mu}) \tag{61}$$

These results are reproduced by the maps,

$$\nabla_{\mu}\phi(x) = \frac{\partial r_a}{\partial x^{\mu}}\nabla^a\phi(r) \tag{62}$$

and

$$\nabla_{\mu}A_{\nu}(x) = \frac{\partial r_a}{\partial x^{\mu}}\frac{\partial r_b}{\partial x^{\nu}}\nabla^a\widehat{A}^b(r) \tag{63}$$

To show this, consider the first of the above relations. Using (14) and the definition (32) of the transverse derivative ∇^a, we obtain,

$$\frac{\partial r_a}{\partial x^{\mu}}\nabla^a\phi = \left(1+s\frac{x^2}{4l^2}\right)^{-2}K_{a\mu}K^{a\sigma}\partial_{\sigma}\phi = \partial_{\mu}\phi \tag{64}$$

that follows on using the identity (18). The cherished result $\nabla_{\mu}\phi = \partial_{\mu}\phi$ is thereby reproduced.

Now using (14), (8), (34) and the identity (18), the R.H.S. of (63) becomes,

$$\frac{\partial r_a}{\partial x^{\mu}}\frac{\partial r_b}{\partial x^{\nu}}\nabla^a\widehat{A}^b(r) = \left(1+s\frac{x^2}{4l^2}\right)^{-2}K_{b\nu}\partial_{\mu}(K^{b\sigma}A_{\sigma}) \tag{65}$$

Putting the structures of the Killing vectors (12), (13) above, we get the R.H.S. of (61), which proves (63).

Now, it is possible to generalise the relations (62) and (63) to include a chain of covariant derivatives. This leads to the following results,

$$\nabla_{\mu}\nabla_{\nu}\phi(x) = \frac{\partial r_a}{\partial x^{\mu}}\frac{\partial r_b}{\partial x^{\nu}}\nabla^a\nabla^b\phi(r) \tag{66}$$

and

$$\nabla_{\mu}\nabla_{\nu}A_{\sigma}(x) = \frac{\partial r_a}{\partial x^{\mu}}\frac{\partial r_b}{\partial x^{\nu}}\frac{\partial r_c}{\partial x^{\sigma}}\nabla^a\nabla^b\widehat{A}^c(r) \tag{67}$$

The generalisation to higher orders is straight forward. The d'Alembertian on the scalar field is next calculated, using (66),

$$\begin{aligned}\Box\phi(x) = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi(x) &= \left(1+s\frac{x^2}{4l^2}\right)^{2}\eta^{\mu\nu}\frac{\partial r_a}{\partial x^{\mu}}\frac{\partial r_b}{\partial x^{\nu}}\nabla^a\nabla^b\phi(r) = \nabla^a\nabla_a\phi(r)\\ &= \nabla^2\phi(r)\end{aligned} \tag{68}$$

Likewise, the d'Alembertian on the vector potential yields, using (67),

$$\Box A_\sigma(x) = g^{\mu\nu}\nabla_\mu\nabla_\nu A_\sigma(x) = \frac{\partial r_c}{\partial x^\sigma}\nabla^2\widehat{A}^c(r) \tag{69}$$

The action for the Yang–Mills theory on the A(dS) space is now defined by first considering the repeated product of the field tensors. Taking (46) and using the transversality of the Killing vectors, we get,

$$\widehat{F}_{abc}\widehat{F}^{abc} = 3sl^2\left(K^{b\mu}K^{c\nu}K_b^\lambda K_c^\rho\right)F_{\mu\nu}F_{\lambda\rho} \tag{70}$$

Finally, using (18), we obtain,

$$\widehat{F}_{abc}\widehat{F}^{abc} = 3sl^2\left(1 + s\frac{x^2}{4l^2}\right)^4 F_{\mu\nu}F^{\mu\nu} \tag{71}$$

Using this identification as well as (23), the actions on the flat space and the A(dS) space are mapped as,

$$S = -\frac{1}{4}\int \mathrm{d}^4x\,\mathrm{Tr}\cdot(F_{\mu\nu}F^{\mu\nu}) = -s\frac{1}{12l^2}\int \mathrm{d}\Omega\,\mathrm{Tr}\cdot(\widehat{F}_{abc}\widehat{F}^{abc}) \tag{72}$$

The lagrangian following from this action is given by

$$\mathcal{L}_\Omega = -s\frac{1}{12l^2}\mathrm{Tr}\cdot(\widehat{F}_{abc}\widehat{F}^{abc}) \tag{73}$$

This completes the construction of the Yang–Mills action which can be taken as the starting point for calculations on the A(dS) space. This action is manifestly gauge invariant under (49). Later on we will discuss an alternative approach to understand this invariance.

Now using the definition of the induced metric (56), it is possible to show that the above action can also be obtained from the standard action defined on the curved space which is taken as,

$$S = -\frac{1}{4}\int \mathrm{d}^4x(\surd - g)g^{\mu\rho}g^{\nu\sigma}\,\mathrm{Tr}\cdot\left(F^D_{\mu\nu}F^D_{\rho\sigma}\right) \tag{74}$$

where

$$F^D_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu] \tag{75}$$

and A_μ has Weyl weight zero for the above action to be conformally invariant. Hence

$$\begin{aligned} F^D_{\mu\nu}F^D_{\rho\sigma} = &\left(\partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu}A_\sigma - \partial_\nu A_\mu + \Gamma^\sigma_{\nu\mu}A_\sigma - \mathrm{i}[A_\mu, A_\nu]\right) \\ &\times\left(\partial_\rho A_\sigma - \Gamma^\lambda_{\rho\sigma}A_\lambda - \partial_\sigma A_\rho + \Gamma^\lambda_{\sigma\rho}A_\lambda - \mathrm{i}[A_\rho, A_\sigma]\right) \end{aligned} \tag{76}$$

Since $\Gamma^\sigma_{\mu\nu}$ is symmetric in two lower indices,

$$F^D_{\mu\nu}F^D_{\rho\sigma} = F_{\mu\nu}F_{\rho\sigma} \tag{77}$$

where $F_{\mu\nu}$ is defined in (34). Taking the explicit form of the induced metric (56) on the A(dS) space, we obtain,

$$\sqrt{(-g)} = \sqrt{(-\det(g_{\mu\nu}))} = \sqrt{\left(-\left(\left(1+s\frac{x^2}{4l^2}\right)^{-8}\right)\det\eta_{\mu\nu}\right)} = \left(1+s\frac{x^2}{4l^2}\right)^{-4} \tag{78}$$

Putting all these values in (74) and using (23), we get

$$S = -\frac{1}{4}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^{4}\left(1+s\frac{x^2}{4l^2}\right)^{-4}\left(1+s\frac{x^2}{4l^2}\right)^{4}\eta^{\mu\rho}\eta^{\nu\sigma}\,\mathrm{Tr}\cdot(F_{\mu\nu}F_{\rho\sigma}) \tag{79}$$

Finally, using (71) we obtain

$$S = -s\frac{1}{12l^2}\int \mathrm{d}\Omega\,\mathrm{Tr}\cdot(\widehat{F}_{abc}\widehat{F}^{abc}) \tag{80}$$

which reproduces (72).

This section is concluded by providing a brief discussion of the gauge fixing condition. On the hypersphere, Adler [22] used the following analogue of the Lorentz condition,

$$L_{ab}\widehat{A}^b - \widehat{A}_a = 0 \tag{81}$$

The same condition is also viable on the A(dS) pseudosphere. Since there is a free index in (81) its connection with the Lorentz gauge is not particularly transparent. A straightforward algebra, however, yields,

$$L_{ab}\widehat{A}^b - \widehat{A}_a = s\frac{1}{l^2}r_a(r_bL^{bc}\widehat{A}_c) \tag{82}$$

so that (81) may be equivalently replaced by

$$r^aL_{ab}\widehat{A}^b = 0 \tag{83}$$

as the pseudospherical analogue of the Lorentz condition. In this form there is no free index. Self-consistency between (81) and (83) is established by contracting the former with r^a and using the transversality condition (9).

In order to prove the identity (82) we simplify its R.H.S. as follows,

$$\begin{aligned}\frac{s}{l^2}r_ar_bL^{bc}\widehat{A}_c &= r_aK^{c\mu}\partial_\mu\widehat{A}_c = \left(r_aK^{c\mu} - r^cK^\mu_a\right)\partial_\mu\widehat{A}_c + r^cK^\mu_a\partial_\mu\widehat{A}_c\\ &= L^c_a\widehat{A}_c + r^cK^\mu_a\partial_\mu\left(K^\sigma_cA_\sigma\right)\end{aligned} \tag{84}$$

where (28) and (29) have been used. Now, exploiting the identity,

$$r^cK^\mu_a\partial_\mu K^\sigma_c = -K^\sigma_a \tag{85}$$

and the transversality condition (9), we obtain,

$$\frac{s}{l^2}r_ar_bL^{bc}\widehat{A}_c = L^c_a\widehat{A}_c - K^\sigma_aA_\sigma = L^c_a\widehat{A}_c - \widehat{A}_a \tag{86}$$

thereby proving (82).

Actually, it is possible to generalise the relation (82) to any vector $\widehat{V}_a$ which is projected to the flat space by

$$\widehat{V}_a = \left(1+s\frac{x^2}{4l^2}\right)^{n}K^\mu_aV_\mu \tag{87}$$

One follows the same steps as before leading to the equivalence,

$$L_{ab}\widehat{V}^b - \widehat{V}_a = \frac{s}{l^2} r_a (r_b L^{bc} \widehat{V}_c) \tag{88}$$

This will be used later. Note that, the mapping for the vector potential (8) corresponds to putting $n = 0$ in (87).

It is simple to prove that the operator in (83) corresponds to the covariant derivative $\nabla_\mu A^\mu$ in the local coordinates. From (63), we obtain,

$$\nabla_\mu A^\mu = g^{\mu\nu}\nabla_\mu A_\nu = \left(1 + s\frac{x^2}{4l^2}\right)^2 \eta^{\mu\nu}\frac{\partial r_a}{\partial x^\mu}\frac{\partial r_b}{\partial x^\nu}\nabla^a\widehat{A}^b = \left(1 + s\frac{x^2}{4l^2}\right)^{-2} K_{a\mu}K_b^\mu \nabla^a \widehat{A}^b \tag{89}$$

Now using the identity (19) and (32) we have,

$$\nabla_\mu A^\mu = s\frac{1}{l^2} r^b L_{ba}\widehat{A}^a \tag{90}$$

This further justifies (83) as the pseudospherical analogue of the Lorentz gauge.

The gauge condition in the form (81) is particularly useful for simplifying the equation of motion. To see this consider the non-abelian equation of motion obtained from (80), by employing the variational principle,[2]

$$\widehat{\mathcal{L}}_{ab}\widehat{F}^{abc} = 0 \tag{91}$$

where the covariantised angular momentum $\widehat{\mathcal{L}}_{ab}$ is defined in (39). This equation is next written in terms of the potential $\widehat{A}_a$. Using the definition of the field tensor (45), the above equation reduces to,

$$\begin{aligned} &L_{ab}L^{ab}\widehat{A}^c - 2L_{ab}L^{ac}\widehat{A}^b - 2\mathrm{i}L_{ab}\{r^a[\widehat{A}^b, \widehat{A}^c]\} - \mathrm{i}L_{ab}\{r^c[\widehat{A}^a, \widehat{A}^b]\} \\ &\quad - \mathrm{i}[r_a\widehat{A}_b - r_b\widehat{A}_a, \widehat{F}^{abc}] = 0 \end{aligned} \tag{92}$$

Now, using (27) we obtain,

$$[L_{ab}, L^{ac}] = -3L_b^c \tag{93}$$

Exploiting this identity in (92) yields,

$$\begin{aligned} &L_{ab}L^{ab}\widehat{A}^c + 6L^{bc}\widehat{A}_b - 2L^{ac}L_{ab}\widehat{A}^b - 2iL_{ab}\{r^a[\widehat{A}^b, \widehat{A}^c]\} - \mathrm{i}L_{ab}\{r^c[\widehat{A}^a, \widehat{A}^b]\} \\ &\quad - \mathrm{i}[r_a\widehat{A}_b - r_b\widehat{A}_a, \widehat{F}^{abc}] = 0 \end{aligned} \tag{94}$$

In terms of the 'covariantised angular momentum' (39) this is written as,

$$P_{ac}\widehat{A}^c = 0 \tag{95}$$

where the wave operator P_{ac} is defined by

$$P_{ac} = \widehat{\mathcal{L}}_{bd}\widehat{\mathcal{L}}^{bd}\eta_{ac} - 6\widehat{\mathcal{L}}_{ac} + 2\widehat{\mathcal{L}}_{ab}\widehat{\mathcal{L}}_c^b \tag{96}$$

Exploiting the angular momentum algebra (27) it is possible to check the following properties of the wave operator,

$$\begin{aligned} &\widehat{\mathcal{L}}^{bc}P_{ca} = P^{bc}\widehat{\mathcal{L}}_{ca} = P_a^b; \\ &r^b P_{ba} = P_{ba}r^a = 0 \end{aligned} \tag{97}$$

[2] See the Appendix A for details.

It is worthwhile to point out the significance of the identities (97). These are related to the gauge invariance of the action (80) under the transformations (40). The action (80) is manifestly gauge invariant under (49). However, there is an alternative way of understanding this invariance which illuminates its connection with the gauge identity. As is known, for gauge theories defined on a flat space, gauge invariance of an action is enforced by a gauge identity; the number of gauge parameters being equal to the number of gauge identities. To see this in the present context, we consider the variation of the action (80) under an arbitrary transformation of the potential, $\delta\widehat{A}_a$:

$$\delta S = \int \mathrm{d}\Omega\,\mathrm{Tr}\cdot[(\delta\widehat{A}_c)(\widehat{\mathcal{L}}_{ab}\widehat{F}^{abc})] = \int \mathrm{d}\Omega\,\mathrm{Tr}\cdot[(\delta\widehat{A}_c)(P^{ca}\widehat{A}_a)] \tag{98}$$

where for simplicity, we have omitted the prefactor $(\frac{s}{12l^2})$. Invariance of the action under an arbitrary $\delta\widehat{A}_a$ yields the equation of motion (91) or (95). If the gauge transformation (40) is now considered, then,

$$\delta S = \frac{s}{l^2}\int \mathrm{d}\Omega\,\mathrm{Tr}\cdot[r^b(\widehat{\mathcal{L}}_{bc}\lambda)(P^{ca}\widehat{A}_a)] \tag{99}$$

Using an integration by parts we find,

$$\delta S = -\frac{s}{l^2}\int \mathrm{d}\Omega\,\mathrm{Tr}\cdot[\lambda\widehat{\mathcal{L}}_{bc}\{r^bP^{ca}\widehat{A}_a\}] \tag{100}$$

Gauge invariance of the action (i.e. $\delta S = 0$) requires that the factor multiplying the gauge parameter λ should vanish identically, i.e. without recourse to any equations of motion. This indeed happens as a consequence of the properties (97),

$$\begin{aligned}\widehat{\mathcal{L}}_{bc}\{r^bP^{ca}\widehat{A}_a\} &= (\widehat{\mathcal{L}}_{bc}r^b)P^{ca}\widehat{A}_a + r^b\widehat{\mathcal{L}}_{bc}P^{ca}\widehat{A}_a = (L_{bc}r^b)P^{ca}\widehat{A}_a + r^bP^a_b\widehat{A}_a\\ &= -(\partial_b r^b)r_cP^{ca}\widehat{A}_a = 0\end{aligned} \tag{101}$$

where the definition (39) of the 'covariantised angular momentum' $\widehat{\mathcal{L}}_{bc}$ has been explicitly used to simplify the first piece. The above identity is usually referred as a gauge identity.

Let us now consider the gauge condition (81). It is equivalently expressed as follows,

$$\widehat{\mathcal{L}}_{ab}\widehat{A}^b - \widehat{A}_a = L_{ab}\widehat{A}^b - \mathrm{i}[r_a\widehat{A}_b - r_b\widehat{A}_a, \widehat{A}^b] - \widehat{A}^a = L_{ab}\widehat{A}^b - \widehat{A}^a = 0 \tag{102}$$

where the transversality condition (9) forces the commutator term to vanish. Now using the gauge condition (102) in (95) we obtain,

$$(\widehat{\mathcal{L}}_{ab}\widehat{\mathcal{L}}^{ab} - 4)\widehat{A}^c = 0 \tag{103}$$

Some comments concerning the equation of motion (103) are now in order. This equation dose not involve the parameter 's' and is identical for both AdS as well as dS spaces. Of course this parameter enters when interactions are introduced. For instance, as discussed later, for Yang–Mills field coupled to fermionic matter, the equation of motion (91) is replaced by

$$\frac{s}{2l^2}\widehat{\mathcal{L}}_{ab}\widehat{F}^{abc} + \hat{j}^c = 0 \tag{104}$$

where $\hat{j}_a$ is the fermionic current. Imposing the gauge condition (81) yields the form analogous to (103),

$$(\widehat{\mathcal{L}}_{ab}\widehat{\mathcal{L}}^{ab}-4)\widehat{A}^c=-2sl^2\hat{j}^c \tag{105}$$

Finally, note that all derivatives occur only through the angular momentum operator which is the correct derivative operator on the A(dS) pseudosphere.

The last point mentioned above is usually violated in other formulations [1,6–11] where the wave operator has a rather complicated structure so that derivatives involve both L_{ab} as well as ∂_a. Hence, to ensure that it dose not go off the pseudosphere, subsidiary conditions are imposed on the field variables. This is now elaborated further.

In the ambient space formulation, starting from the Casimir operator and using the infinitesimal generators, the Casimir eigenvalue equation and some algebric properties, the field equation for a massless abelian vector field in A(dS) space is found to be [10],[3]

$$(sl^2\nabla^2-2)\widehat{A}_a+2\frac{s}{l^2}r_ar^bL_{bc}\widehat{A}^c-sl^2\nabla_a(\partial^b\widehat{A}_b)=0 \tag{106}$$

A similar equation is also given in [6] which exploits the irreducible representations of the kinamatical A(dS) groups or using the triplet formalism [7]. The subsidiary (or divergenceless) condition $\partial^b\widehat{A}_b=0$ is next imposed to eliminate the last term that explicitly involves the derivative ∂^b. For comparing (106) with our result, it is now necessary to identify ∇^2 with $L_{ab}L^{ab}$. To see this we use the expression for the angular momentum operator (32), to derive,

$$\frac{s}{2l^2}L_{ab}L^{ab}=\nabla^2+\frac{s}{l^2}(\nabla^br^a)(r_a\nabla_b) \tag{107}$$

where we have used the results, $r_a\nabla^a=0$ and $\nabla^ar_a=4$. Then,

$$\begin{aligned}\nabla^2-\frac{s}{2l^2}L_{ab}L^{ab}&=-\frac{s}{l^2}(\nabla^br^a)(r_a\nabla_b)=\left(1+s\frac{x^2}{4l^2}\right)^2r_a(\partial_\mu r^a)\partial^\mu\\&=\left(1+s\frac{x^2}{4l^2}\right)^2\frac{1}{2}\partial_\mu(r_ar^a)\partial^\mu=0\end{aligned} \tag{108}$$

where the constancy of the A(dS) pseudosphere $r_ar^a=sl^2$ is used to get the vanishing result. Hence, we obtain the identification,

$$\nabla^2=\frac{s}{2l^2}L_{ab}L^{ab} \tag{109}$$

The Lorentz gauge fixing condition in the form (83) is now used to eliminate the second term in (106). Finally, putting (109) in (106) we find,

$$(L_{bc}L^{bc}-4)\widehat{A}_a=0 \tag{110}$$

which exactly corresponds to the abelian version of (103). It might be recalled that (103) was also obtained by using the Lorentz gauge, albeit in the form (81). Incidentally it is also possible to rewrite (103) using the covariantised transverse derivative (42). First, we express (39) in terms of this derivative as,

$$\widehat{\mathcal{L}}_{ab}=r_a\tilde{\nabla}_b-r_b\tilde{\nabla}_a \tag{111}$$

[3] This paper discusses only the de Sitter example while Refs. [6,7] consider the anti-de Sitter case. There is no collective discussion.

Taking its repeated product yields,

$$\frac{s}{2l^2}\widehat{\mathcal{L}}_{ab}\widehat{\mathcal{L}}^{ab} = \tilde{\nabla}_a\tilde{\nabla}^a + \frac{s}{l^2}r_a(\tilde{\nabla}_b r^a)\tilde{\nabla}^b = \tilde{\nabla}_a\tilde{\nabla}^a + \frac{s}{l^2}r_a(\nabla_b r^a)\nabla^b = \tilde{\nabla}_a\tilde{\nabla}^a = \tilde{\nabla}^2 \qquad (112)$$

where use has been made of (108). Hence the following form of (103) is obtained,

$$(sl^2\tilde{\nabla}^2 - 2)\widehat{A}_a = 0 \qquad (113)$$

which can be interpreted as the non-abelian extension of (106), subject to subsidiary and Lorentz gauge conditions.

We conclude this section by discussing two issues; the possibilities of hyperbolic A(dS) space as infrared regulators and secondly, the A(dS)/CFT correspondence in our approach.

It was shown in [12] that a space of constant negative curvature like the AdS space provided a good infrared regulator for euclidean quantum field theory. The method used there was also based on stereographic projection. Hence it is simple to relate the findings of [12] with our analysis. In fact the AdS gauge field equation of motion, which is the starting point in [12], just corresponds to the $s = 1$ version of (105). There was an arbitrariness in the general solution of the Green function which was appropriately tuned to improve the asymptotic behaviour. Since the de Sitter case implies $s = -1$ in (105), these arguments go through and one expects that the infrared properties also improve here. This is presumably related to the fact that time slices of the de Sitter space can be chosen to make the spatial volume finite thereby strongly affecting the infrared behaviour. Indeed it is precisely this property of the boundedness of volume that makes euclidean QED on the sphere manifestly infrared-finite [22].

Recently, there have been several studies [13,14] to extend the AdS/CFT correspondence [15] to include the de Sitter space. While it is unclear whether the hypothesised dS/CFT correspondence is on the same footing as the AdS/CFT one, it is possible to show their equivalence at the algebric level. The point is that the isometry group $SO(2,n)$ (for AdS_{n+1}) or $SO(1,n+1)$ (for dS_{n+1}) acts on the boundary as the conformal group acting on Minkowski (euclidean) space. The same geometry as defined by (1) is used. Light cone coordinates are then introduced to define the 'projective boundary' of A(dS) space. The purported results follow by considering the action of the group – say $SO(1,n+1)$ – on the boundary points. We refer to [16] for relevant technical details.

4. Stereographic projections of the Dirac equation on A(dS) space

In order to discuss the stereographic projection of the Dirac equation it is first necessary to introduce the various Dirac matrices. In the ordinary $D = 4$ Minkowski space, these matrices satisfy,

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}; \quad \{\gamma_\mu, \gamma_4\} = 0 \qquad (114)$$

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0; \quad \gamma_4^\dagger = \gamma_4; \quad \gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3; \quad (\gamma_4)^2 = 1 \qquad (115)$$

Now we define the gamma matrices on the de Sitter (dS) space as,

$$\Gamma_\mu = \gamma_\mu\gamma_4 \qquad (116)$$

$$\Gamma_4 = \gamma_4 \qquad (117)$$

while, for the anti-de Sitter (AdS) space as,

$$\Gamma_\mu = \mathrm{i}\gamma_\mu\gamma_4 \tag{118}$$

$$\Gamma_4 = \gamma_4 \tag{119}$$

These gamma matrices are defined so that, in either case, they obey the following properties,

$$\{\Gamma_a, \Gamma_b\} = 2s\eta_{ab} \tag{120}$$

and

$$(r\cdot\Gamma)^2 = l^2 \tag{121}$$

In order that the current satisfies a transversality condition like (9), we take its form on A(dS) space as,

$$\hat{j}_a = p\frac{1}{2l}\overline{\widehat{\Psi}}[\Gamma\cdot r, \Gamma_a]\widehat{\Psi} \tag{122}$$

so that $r^a\hat{j}_a = 0$. Here, $p = 1$ for dS space and $p = -i$ for AdS space. The field, $\widehat{\Psi}$ is the Dirac spinor on the A(dS) space which is mapped to the Dirac spinor Ψ on the Minkowski space through the following relation,

$$\widehat{\Psi} = \left(1 - \frac{x^2}{4l^2}\right)\left(1 - \frac{x^\mu}{2l}\gamma_\mu\right)\Psi \tag{123}$$

for dS space. For AdS space the corresponding map is,

$$\widehat{\Psi} = \left(1 + \frac{x^2}{4l^2}\right)\left(1 - \mathrm{i}\frac{x^\mu}{2l}\gamma_\mu\right)\Psi \tag{124}$$

From these relations, we define the adjoint spinor as,

$$\overline{\widehat{\Psi}} = \widehat{\Psi}^\dagger\Gamma_0 = \left(1 - \frac{x^2}{4l^2}\right)\overline{\Psi}\left(1 + \frac{x^\mu}{2l}\gamma_\mu\right) \tag{125}$$

for dS space and for AdS space as,

$$\overline{\widehat{\Psi}} = \widehat{\Psi}^\dagger\Gamma_0\Gamma_4 = \left(1 + \frac{x^2}{4l^2}\right)\overline{\Psi}\left(1 + \mathrm{i}\frac{x^\mu}{2l}\gamma_\mu\right) \tag{126}$$

This completes the stereographic mapping of the Dirac spinor. We now use these expressions to prove,

$$\hat{j}_a = \left(1 + s\frac{x^2}{4l^2}\right)^2 K^\mu_a j_\mu \tag{127}$$

which is the exact analog of (8), apart from the conformal factor.

Here the explicit calculation for dS space is given. Calculations for AdS space are similar. For $a = \mu$ we have from (122),

$$\hat{j}_\mu = \frac{1}{2l}\left(1-\frac{x^2}{4l^2}\right)^2\left(r^\alpha[\gamma_\alpha,\gamma_\mu] - \frac{r^\alpha x^\lambda}{2l}[\gamma_\alpha,\gamma_\mu]\gamma_\lambda + 2r^4\gamma_\mu - \frac{2r^4x^\lambda}{2l}\gamma_\mu\gamma_\lambda \right.$$
$$\left. + \frac{r^\alpha x^\nu}{2l}\gamma_\nu[\gamma_\alpha,\gamma_\mu] - \frac{r^\alpha x^\lambda x^\nu}{4l^2}\gamma_\nu[\gamma_\alpha,\gamma_\mu]\gamma_\lambda + \frac{2r^4x^\nu}{2l}\gamma_\nu\gamma_\mu - \frac{2r^4x^\lambda x^\nu}{4l^2}\gamma_\nu\gamma_\mu\gamma_\lambda\right) \tag{128}$$

Using the identity,

$$\gamma_\mu\gamma_\nu\gamma_\alpha = \eta_{\mu\nu}\gamma_\alpha - \eta_{\mu\alpha}\gamma_\nu + \eta_{\nu\alpha}\gamma_\mu - \mathrm{i}\epsilon_{\mu\nu\alpha\lambda}\gamma^\lambda\gamma_4 \tag{129}$$

and the properties of gamma matrices (114) and (115) we get,

$$\hat{j}_\mu = \left(1-\frac{x^2}{4l^2}\right)^2 K^\nu_\mu j_\nu \tag{130}$$

Similarly for $a = 4$ one can show,

$$\hat{j}_4 = \left(1-\frac{x^2}{4l^2}\right)^2 K^\mu_4 j_\mu \tag{131}$$

These two can be written as,

$$\hat{j}_a = \left(1-\frac{x^2}{4l^2}\right)^2 K^\mu_a j_\mu \tag{132}$$

For AdS space this will be,

$$\hat{j}_a = \left(1+\frac{x^2}{4l^2}\right)^2 K^\mu_a j_\mu \tag{133}$$

This proves Eq. (127). The implication of the conformal factor in (127) becomes evident when we introduce coupling with the gauge fields. This factor is necessary to ensure the form invariance of the interaction term in the action,

$$\int \mathrm{d}\Omega(\hat{j}_a\widehat{A}^a) = \int \mathrm{d}^4x(j_\mu A^\mu) \tag{134}$$

This is verified by an explicit use of the various maps. Indeed in D-dimensions, the map (127) reads as,

$$\hat{j}_a = \left(1+s\frac{x^2}{4l^2}\right)^{D-2} K^\mu_a j_\mu \tag{135}$$

so that form invariance of the interaction term in the action as illustrated in (134) is valid in any dimensions. The inverse map, computed from (127) by contracting with the Killing vector and using the identity (18), is given by

$$j_\mu = \left(1+s\frac{x^2}{4l^2}\right)^{-4} K_{a\mu}\hat{j}^a = \left(1+s\frac{x^2}{4l^2}\right)^{-2}\frac{\partial r_a}{\partial x^\mu}\hat{j}^a \tag{136}$$

We have seen that the vector fields are mapped by the conformal Killing vectors. So, we can expect that the Dirac spinors are mapped by the conformal Killing spinors. To show this we define a transformation matrix W such that,

$$W = \left(1 + \frac{x^\mu}{2l}\gamma_\mu\right) \tag{137}$$

for dS space and for AdS space,

$$W = \left(1 + \mathrm{i}\frac{x^\mu}{2l}\gamma_\mu\right) \tag{138}$$

The adjoint of 'W', following (125), is given by

$$\overline{W} = W^\dagger \Gamma_0 = \Gamma_0\left(1 - \frac{x^\mu}{2l}\gamma_\mu\right) \tag{139}$$

for dS space. For AdS space the corresponding definition follows from (126),

$$\overline{W} = W^\dagger \Gamma_0 \Gamma_4 = \Gamma_0 \Gamma_4\left(1 - \mathrm{i}\frac{x^\mu}{2l}\gamma_\mu\right) \tag{140}$$

Therefore, for dS space we have,

$$\overline{W}W = \Gamma_0\left(1 - \frac{x^2}{4l^2}\right) \tag{141}$$

and the corresponding relation for AdS space is,

$$\overline{W}W = \Gamma_0\Gamma_4\left(1 + \frac{x^2}{4l^2}\right) \tag{142}$$

Using the above equations, the fermion maps (123) and (124) are written collectively as,

$$\Psi = \left(1 + s\frac{x^2}{4l^2}\right)^{-2} W\widehat{\Psi} \tag{143}$$

The maps for the adjoint spinors, on the other hand, are different. For dS space (125) becomes,

$$\overline{\Psi} = -\left(1 - \frac{x^2}{4l^2}\right)^{-2} \widehat{\overline{\Psi}}\Gamma_0\overline{W} \tag{144}$$

while for AdS space, (126) simplifies to,

$$\overline{\Psi} = \left(1 + \frac{x^2}{4l^2}\right)^{-2} \widehat{\overline{\Psi}}\Gamma_0\Gamma_4\overline{W} \tag{145}$$

The mapping (143) of fermions from A(dS) space to flat space has now been expressed in terms of 'W' which is the conformal Killing spinor satisfying the equation,

$$\partial_\mu W = \frac{1}{4}\gamma_\mu(\gamma^\sigma\partial_\sigma W) \tag{146}$$

This is analogous to (10) that defines the conformal Killing vector. The above relation yields a general definition for a conformal Killing spinor. Similar relations have appeared previously in the literature [23].

It is known that [18] there is a bilinear map connecting the conformal Killing vectors with the conformal Killing spinors. To see this we have to write (122) in a different form which is given below. Using (120) and (121) one can show that

$$[\Gamma\cdot r,\Gamma_a]=-2\left(\eta_a^b-s\frac{r_a r^b}{l^2}\right)\Gamma_b(\Gamma\cdot r) \tag{147}$$

Therefore, the current (122) on A(dS) space is also expressed as,

$$\hat{j}_a=-p\frac{1}{l}\overline{\widehat{\Psi}}\left(\eta_a^b-s\frac{r_a r^b}{l^2}\right)\Gamma_b(\Gamma\cdot r)\widehat{\Psi}=\frac{-p}{l}\overline{\widehat{\Psi}}\theta_a^b\Gamma_b\Gamma\cdot r\widehat{\Psi} \tag{148}$$

Now with the help of (19), the above equation is written as,

$$\hat{j}_a=-p\frac{1}{l}\left(1+s\frac{x^2}{4l^2}\right)^{-2}\overline{\widehat{\Psi}}K_{a\mu}K^{b\mu}\Gamma_b(\Gamma\cdot r)\widehat{\Psi} \tag{149}$$

Therefore, by (127) we have,

$$-p\frac{1}{l}\left(1+s\frac{x^2}{4l^2}\right)^{-4}\overline{\widehat{\Psi}}K_{a\mu}K^{b\mu}\Gamma_b(\Gamma\cdot r)\widehat{\Psi}=\overline{\Psi}K_a^{\mu}\gamma_\mu\Psi \tag{150}$$

Using the fermionic spinor maps (123) and (125) (for dS space), the above can be written as,

$$\frac{1}{l}\left(1-\frac{x^2}{4l^2}\right)^{-2}WK_{b\mu}\Gamma^b(\Gamma\cdot r)\Gamma_0\overline{W}=\gamma_\mu \tag{151}$$

Multiplying '$\overline{W}$' from left and then 'W' from right on both sides of the above equation and using (141) we get,

$$\overline{W}\gamma_\mu W=\frac{1}{l}K_{b\mu}(\Gamma_0\Gamma^b)(\Gamma\cdot r) \tag{152}$$

for dS space. For AdS space, the calculation is similar and yields,

$$\overline{W}\gamma_\mu W=\frac{\mathrm{i}}{l}K_{b\mu}(\Gamma_0\Gamma_4\Gamma^b)(\Gamma\cdot r) \tag{153}$$

These are the cherished bilinear relations between the Killing spinors and the Killing vectors.

Let us next consider the definition of the axial current. The analogoue of γ_4 (chirality operator on Minkowski space) is given here by $\frac{1}{l}\Gamma\cdot r$ since it satisfies,

$$\left(\frac{1}{l}\Gamma\cdot r\right)^2=1 \tag{154}$$

and the anti-commutator,

$$\left\{\frac{\Gamma\cdot r}{l},[\Gamma\cdot r,\Gamma_a]\right\}=0 \tag{155}$$

The projection operators are given by

$$P_{\pm} = \frac{1 \pm \frac{\Gamma \cdot r}{l}}{2}, ; \quad P_{\pm}^2 = P_{\pm}, ; \quad P_+ P_- = 0 \tag{156}$$

Hence the axial current is defined as,

$$\hat{j}_{a5} = -p \frac{1}{2l^2} \overline{\widehat{\Psi}} [\Gamma \cdot r, \Gamma_a] (\Gamma \cdot r) \widehat{\Psi} \tag{157}$$

This also satisfies the transversality condition $r^a \hat{j}_{a5} = 0$. As was shown for the vector current, one can prove,

$$\hat{j}_{a5} = \left(1 + s\frac{x^2}{4l^2}\right)^2 K_a^{\mu} j_{\mu 5} \tag{158}$$

where $j_{\mu 5} = \widehat{\Psi} \gamma_{\mu} \gamma_4 \Psi$ is the axial current on the flat space. This can also be written (similar to (148)) as,

$$\hat{j}_{a5} = -p \overline{\widehat{\Psi}} \left(\eta_a^b - s\frac{r_a r^b}{l^2}\right) \Gamma_b \widehat{\Psi} = -\overline{\widehat{\Psi}} \theta_a^b \Gamma_b \widehat{\Psi} \tag{159}$$

Now we will write the Dirac operator and the Dirac equation on A(dS) space. For this we define,

$$S_{ab} = \frac{1}{4} [\Gamma_a, \Gamma_b] \tag{160}$$

Using the definition for S_{ab} given above, one can show the algebric relation,

$$S_{ab} L^{ab} = -\frac{1}{4} [\gamma_{\mu}, \gamma_{\nu}] L^{\mu\nu} + \gamma_{\mu} L^{4\mu} \tag{161}$$

for dS space. The corresponding relation for AdS space is,

$$S_{ab} L^{ab} = \frac{1}{4} [\gamma_{\mu}, \gamma_{\nu}] L^{\mu\nu} - \mathrm{i} \gamma_{\mu} L^{4\mu} \tag{162}$$

These relations are collectively expressed as,

$$S_{ab} L^{ab} = \frac{s}{4} [\gamma_{\mu}, \gamma_{\nu}] L^{\mu\nu} + p \gamma_{\mu} L^{4\mu} \tag{163}$$

Again using (12), (13) and (28) we get the form of the angular momentum operator on the flat space as,

$$L_{\mu\nu} = x_{\mu} \partial_{\nu} - x_{\nu} \partial_{\mu} \tag{164}$$

$$L_{4\mu} = -sl \left(1 - s\frac{x^2}{4l^2}\right) \partial_{\mu} - \frac{1}{2l} x_{\mu} x^{\nu} \partial_{\nu} \tag{165}$$

Using the fermion maps (123), (125) and (161) together with the expression for the angular momentum operator given above we obtain,

$$\overline{\widehat{\Psi}} (S_{ab} L^{ab} + 2) \widehat{\Psi} = l \left(1 - \frac{x^2}{4l^2}\right)^4 \overline{\Psi} \gamma^{\mu} \partial_{\mu} \Psi \tag{166}$$

for dS space. The corresponding calculations for AdS space are exactly the same. For this one can show,

$$\overline{\widehat{\Psi}}(-S_{ab}L^{ab}+2)\widehat{\Psi} = l\left(1+\frac{x^2}{4l^2}\right)^4\overline{\Psi}\gamma^\mu\partial_\mu\Psi \tag{167}$$

So, in general we can write,

$$\overline{\widehat{\Psi}}(-sS_{ab}L^{ab}+2)\widehat{\Psi} = l\left(1+s\frac{x^2}{4l^2}\right)^4\overline{\Psi}\gamma^\mu\partial_\mu\Psi \tag{168}$$

This enables us to convert the Dirac operator into the pseudospherical operator by the map,

$$l\gamma^\mu\partial_\mu \to -sS_{ab}L^{ab}+2 \tag{169}$$

The Dirac action on the flat Minkowski space is given by

$$S = -\mathrm{i}\int \mathrm{d}^4x\,\overline{\Psi}\gamma^\mu\partial_\mu\Psi \tag{170}$$

Using the map (23) for the measure and (168), we obtain the following action for A(dS) space by a projection of the above flat action,

$$S = -\frac{\mathrm{i}}{l}\int \mathrm{d}\Omega\,\overline{\widehat{\Psi}}(-sS_{ab}L^{ab}+2)\widehat{\Psi} \tag{171}$$

It is possible to show that the above action, exactly in analogy to the case of the gauge field, can be obtained from the standard action defined on the curved space which is taken as,

$$S = \int \mathrm{d}^4x(\sqrt{}-g)\overline{\Psi}_c e^{\mu\nu}\gamma_\nu\nabla_\mu\Psi_c \tag{172}$$

where 'Ψ_c' is the Dirac spinor on the curved space and is connected to the flat space Dirac spinor through the conformal factor $\left(1+s\frac{x^2}{4l^2}\right)^{\frac{3}{2}}$, which is required for the above action to be conformally invariant. The vielbein $e_{\mu\nu}$ is related to the flat Minkowski metric by the following relation,

$$e_{\mu\nu} = \left(1+s\frac{x^2}{4l^2}\right)^{-1}\eta_{\mu\nu} \tag{173}$$

The covariant derivative ∇_μ for a spinor is,

$$\nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu\alpha\beta}\sigma^{\alpha\beta} \tag{174}$$

where the spin connection is defined as,

$$\omega_{\mu\alpha\beta} = \frac{1}{2}(C_{\mu,\alpha\beta} - C_{\alpha,\beta\mu} - C_{\beta,\mu\alpha}) \tag{175}$$

$$C^\nu_{,\alpha\beta} = \left(\partial_\alpha e^\nu_\beta - \partial_\beta e^\nu_\alpha\right) \tag{176}$$

and

$$\sigma^{\alpha\beta} = \frac{1}{4}[\gamma^\alpha,\gamma^\beta] \tag{177}$$

Now using (23), (56) and (173) and the transformation relation $\Psi_c = (1 + s\frac{x^2}{4l^2})^{\frac{3}{2}}\Psi$, the first part of the action (172) involving the ordinary derivative can be written as,

$$
\begin{aligned}
S_1 &= -\mathrm{i}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^4\left(1+s\frac{x^2}{4l^2}\right)^{-4}\left(1+s\frac{x^2}{4l^2}\right)^{\frac{3}{2}}\overline{\Psi}\left(1+s\frac{x^2}{4l^2}\right)\eta^{\mu\nu}\gamma_\nu\partial_\mu\left[\left(1+s\frac{x^2}{4l^2}\right)^{\frac{3}{2}}\Psi\right] \\
&= -\mathrm{i}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^4\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{3\mathrm{i}s}{4l^2}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^3\overline{\Psi}\gamma^\mu x_\mu\Psi
\end{aligned}
\tag{178}
$$

For the second part we have to first calculate the form of the spin connection. Using (173) and (176) we have,

$$
\begin{aligned}
C_{\mu,\alpha\beta} &= e_{\mu\nu}C^{\nu}_{\ ,\alpha\beta} = e_{\mu\nu}[\partial_\alpha(g^{\nu\lambda}e_{\beta\lambda}) - \partial_\beta(g^{\nu\lambda}e_{\alpha\lambda})] \\
&= \frac{s}{2l^2}\left(1+s\frac{x^2}{4l^2}\right)^{-1}(x_\alpha\eta_{\beta\mu} - x_\beta\eta_{\alpha\mu})
\end{aligned}
\tag{179}
$$

Therefore, using the definition for the spin connection (175), we obtain,

$$
\omega_{\mu\alpha\beta} = \frac{s}{2l^2}\left(1+s\frac{x^2}{4l^2}\right)^{-1}(x_\alpha\eta_{\beta\mu} - x_\beta\eta_{\alpha\mu}) \tag{180}
$$

Putting all the results in the second part of the action we get,

$$
S_2 = -\frac{\mathrm{i}s}{16l^2}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^3\overline{\Psi}(x_\alpha\eta_{\beta\mu} - x_\beta\eta_{\alpha\mu})\gamma^\mu[\gamma^\alpha,\gamma^\beta]\Psi \tag{181}
$$

Now, using the previous identity (129), we find,

$$
(x_\alpha\eta_{\beta\mu} - x_\beta\eta_{\alpha\mu})\gamma^\mu[\gamma^\alpha,\gamma^\beta] = -12x_\mu\gamma^\mu \tag{182}
$$

Putting this result in (181) we obtain,

$$
S_2 = \frac{3\mathrm{i}s}{4l^2}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^3\overline{\Psi}\gamma^\mu x_\mu\Psi \tag{183}
$$

Therefore, adding these two parts (178) and (183) of the action (172) we get,

$$
S = S_1 + S_2 = -\mathrm{i}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^4\overline{\Psi}\gamma^\mu\partial_\mu\Psi \tag{184}
$$

Finally using (168) we obtain the cherished form,

$$
S = -\frac{\mathrm{i}}{l}\int \mathrm{d}\Omega\,\overline{\widehat{\Psi}}(-sS_{ab}L^{ab} + 2)\widehat{\Psi} \tag{185}
$$

which reproduces (171).

So the Dirac equation on A(dS) space is given by

$$
(-sS_{ab}L^{ab} + 2)\widehat{\Psi} = 0 \tag{186}
$$

For the hypersphere an analogous relation is given in [22,24].

For general D-dimensions the Dirac equation on A(dS) space can be written as:

$$\left(-sS_{ab}L^{ab} + \frac{D}{2}\right)\widehat{\Psi} = 0 \tag{187}$$

For $D = 4$ we get back equation(186).

In presence of fermionic matter the flat space Yang–Mills action has the form,

$$S = \int \mathrm{d}^4x\left[-\frac{1}{4}\mathrm{Tr}\cdot(F_{\mu\nu}F^{\mu\nu}) - \mathrm{i}\overline{\Psi}\gamma^{\mu}(\partial_{\mu} - \mathrm{i}eA_{\mu})\Psi\right] \tag{188}$$

Using appropriate expressions for each piece, the stereographically projected action on the A(dS) space becomes,

$$\widehat{S} = \int \mathrm{d}\Omega\left[-\frac{s}{12l^2}\mathrm{Tr}\cdot(\widehat{F}_{abc}\widehat{F}^{abc}) - \frac{\mathrm{i}}{l}\widehat{\overline{\Psi}}(-sS_{ab}L^{ab} + 2)\widehat{\Psi} - e\widehat{A}_a\hat{j}^a\right] \tag{189}$$

where $\hat{j}^a$ is defined in (122) and we have used (127) to project the interaction term. Hence, in the presence of gauge fields, Eq. (186) becomes,

$$\left[\frac{\mathrm{i}s}{l}(S_{ab})_{ij}\eta^m_n L^{ab} - \frac{2\mathrm{i}}{l}\eta^m_n\delta_{ij} + \frac{2ep}{l}(S_{ab})_{ij}r^b(\widehat{A^{\lambda a}})(T^{\lambda})^m_n\right]\widehat{\Psi}_{jm} = 0 \tag{190}$$

By looking at the Dirac Eq. (186) or (187) one might be tempted to interpret the numerical factor as a mass term on A(dS) space. But this is not true. It is seen from (169) that this Dirac operator was obtained from a projection of the massless Dirac operator on the flat space. Also, it is equivalent to the massless Dirac operator (172) defined on a curved background.

Yet another way is to use chiral invariance. In the flat space the mass term breaks this invariance in the sence that the anti-commuting property $\{\gamma^{\mu}\partial_{\mu}, \gamma_5\} = 0$ of the massless Dirac operator no longer holds. In A(dS) space the analogue of γ_5 is $\frac{\Gamma\cdot r}{l}$, as already discussed. We now compute the anticommutator of the Dirac operator appearing in (187) with $\frac{\Gamma\cdot r}{l}$ and show that it vanishes, thereby reconferming that the numerical factor should not be interpreted as a mass term. The de Sitter space calculation is given below. The calculation for AdS space is exactly similar. Now,

$$\left\{S\cdot L + \frac{D}{2}, \frac{\Gamma\cdot r}{l}\right\}\phi = \frac{1}{l}\{S\cdot L, \Gamma\cdot r\}\phi + D\frac{\Gamma\cdot r}{l}\phi \tag{191}$$

The first term of the above equation yields,

$$\frac{1}{l}\{S\cdot L(\Gamma\cdot r\phi) + \Gamma\cdot rS\cdot L\phi\} = \frac{1}{l}\{S_{ab}(L^{ab}\Gamma\cdot r)\phi + S_{ab}\Gamma\cdot rL^{ab}\phi + \Gamma\cdot rS\cdot L\phi\} \tag{192}$$

Now the last two terms of the above cancel each other, which is shown below.

$$\begin{aligned} S_{ab}\Gamma\cdot rL^{ab} + \Gamma\cdot rS\cdot L &= \frac{1}{4}[\Gamma_a, \Gamma_b]\Gamma\cdot rL^{ab} + \frac{1}{4}\Gamma\cdot r[\Gamma_a, \Gamma_b]L^{ab} \\ &= \frac{1}{2}\Gamma_a\Gamma_b\Gamma\cdot r(r^a\partial^b - r^b\partial^a) + \frac{1}{2}\Gamma\cdot r\Gamma_a\Gamma_b(r^a\partial^b - r^b\partial^a) \\ &= \frac{1}{2}[(\Gamma\cdot r)\Gamma_a(\Gamma\cdot r)\partial^a - \Gamma_a(\Gamma\cdot r)^2\partial^a + (\Gamma\cdot r)^2\Gamma_a\partial^a \\ &\quad - (\Gamma\cdot r)\Gamma_a(\Gamma\cdot r)\partial^a] = 0 \end{aligned} \tag{193}$$

where we have used (154) in the last line.

Now,

$$S_{ab}L^{ab}(\Gamma\cdot r)=S_{\mu\nu}L^{\mu\nu}(\Gamma\cdot r)+S_{D\mu}L^{D\mu}(\Gamma\cdot r)=\frac{1}{2}[\gamma_\nu\gamma_\mu L^{\mu\nu}(\Gamma\cdot r)-\gamma_\mu L^{D\mu}(\Gamma\cdot r)]$$
$$=\frac{1}{2}[\gamma_\nu\gamma_\mu\gamma_\alpha\gamma_D L^{\mu\nu}r^\alpha+\gamma_\nu\gamma_\mu\gamma_D L^{\mu\nu}r^D+\gamma_\mu\gamma_\nu\gamma_D L^{D\mu}r^\nu+\gamma_\mu\gamma_D L^{D\mu}r^D] \tag{194}$$

Using the expressions for $L^{\mu\nu}$(164), $L^{D\mu}$(165), r^μ and r^D(2) we have,

$$\begin{aligned} L^{\mu\nu}r^\alpha &= \frac{1}{1-\frac{x^2}{4l^2}}(x^\mu\eta^{\nu\alpha}-x^\nu\eta^{\mu\alpha}) \\ L^{\mu\nu}r^D &= 0 \\ L^{D\mu}r^\nu &= -\frac{l\left(1+\frac{x^2}{4l^2}\right)}{1-\frac{x^2}{4l^2}}\eta^{\mu\nu} \\ L^{D\mu}r^D &= \frac{x^\mu}{1-\frac{x^2}{4l^2}} \end{aligned} \tag{195}$$

Therefore,

$$S_{ab}L^{ab}(\Gamma\cdot r)=\frac{(1-D)x^\mu\gamma_\mu\gamma_D}{1-\frac{x^2}{4l^2}}+D\frac{l\left(1+\frac{x^2}{4l^2}\right)}{1-\frac{x^2}{4l^2}}\gamma_D-\frac{x^\mu}{1-\frac{x^2}{4l^2}}\gamma_\mu\gamma_D$$
$$=(1-D)r^\mu\Gamma_\mu-Dr^D\Gamma_D-r^\mu\Gamma_\mu=-D(\Gamma\cdot r) \tag{196}$$

Using all these results, we obtain,

$$\left\{S\cdot L+\frac{D}{2},\frac{\Gamma\cdot r}{l}\right\}=0 \tag{197}$$

For AdS space this will be $\{-S\cdot L+\frac{D}{2},\frac{\Gamma\cdot r}{l}\}=0$. So, in general we obtain,

$$\left\{-sS\cdot L+\frac{D}{2},\frac{\Gamma\cdot r}{l}\right\}=0 \tag{198}$$

This shows that $\frac{D}{2}$ cannot be the mass term and the free Dirac operator for massless fermion on general D-dimensional A(dS) space is $\frac{1}{l}(-sS\cdot L+\frac{D}{2})$.

Now if we include a mass term, then the massive Dirac action on A(dS) space for 4-dimensions can be written as,

$$S=\int \mathrm{d}\Omega\overline{\widehat{\Psi}}\left[-\frac{\mathrm{i}}{l}(-sS\cdot L+2)+\nu\right]\widehat{\Psi} \tag{199}$$

where 'ν' plays the role of mass of fermion on A(dS) space. 'ν' can be determined by our projection method. The mass term of the flat space action is given by $m\overline{\Psi}\Psi$. Using the projection for the spinor field (143)–(145) we obtain,

$$m\overline{\Psi}\Psi=m\left(1+s\frac{x^2}{4l^2}\right)^{-3}\overline{\widehat{\Psi}}\widehat{\Psi} \tag{200}$$

So the Dirac action on A(dS) space for the mass sector is,

$$S_m = \int \mathrm{d}\Omega \left(1 + s\frac{x^2}{4l^2}\right)^4 \cdot m \left(1 + s\frac{x^2}{4l^2}\right)^{-3} \overline{\widehat{\Psi}}\widehat{\Psi} = \int \mathrm{d}\Omega m \left(1 + s\frac{x^2}{4l^2}\right) \overline{\widehat{\Psi}}\widehat{\Psi} \tag{201}$$

Therefore the mass of the fermion on A(dS) space is given by

$$v = m\left(1 + s\frac{x^2}{4l^2}\right) \tag{202}$$

In the flat space limit $l \to \infty$ we get back the usual mass.

It may be mentioned that so far the analysis has been basically classical. Extension to quantum field theory is quite nontrivial. In the next section we will take up this issue in some detail where an explicit calculation of the quantum anomaly is presented and its equivalence is established with our stereographic projection approach.

There is another point to consider. Any quantum effect on the A(dS) space need not have a counterpart on the flat space. For instance, a quantum field theory on dS space will reveal the analogue of Hawking radiation because of the cosmological horizon. However, it appears highly unlikely that this physical effect would naturally connect to a corresponding feature of the projected theory on flat space.

As a simple illustration consider the factorisation of the wave operator for Bose particles in terms of the fermionic wave operator that holds in flat space,

$$(i\gamma \cdot \partial)(i\gamma \cdot \partial) = -\Box \tag{203}$$

This dose not have an analogue on the A(dS) space. As we have shown, here the Bose operator (in four dimensions) is $(L_{ab}L^{ab} - 4)$ while the Fermi operator is $(-sS_{ab}L^{ab} + 2)$. It is now possible to show [12,22],

$$(-sS_{ab}L^{ab} + 2)(-sS_{ab}L^{ab} + 1) = -\frac{1}{2}(L_{ab}L^{ab} - 4) \tag{204}$$

The simple factorisation in flat space therefore dose not hold in the A(dS) space.

5. Chiral anomalies on A(dS) space

In this section we discuss the structure of chiral anomalies on A(dS) space. First, an explicit evaluation of the axial U(1) anomaly is computed in the path integral approach. This result is next reproduced by an appropriate stereographic map of the usual Adler-Bell-Jackiw [25] flat space expression. This shows that the methods developed here are meaningful for considering quantum effects. Finally, we obtain the non-abelian chiral anomalies by our projection technique.

5.1. The axial anomaly in the path integral formalism

In a series of papers, Fujikawa [26] has shown how the chiral anomalies encountered in perturbation theory may be derived in a path integral framework. In this approach the anomalous behaviour of Ward-Takahashi identities is traced to the Jacobian factor arising from the noninvariance of the path integral measure under chiral transformations. Here we compute the chiral anomaly in the A(dS) space using this method. We follow the approach of [27] where the analysis has been done on the sphere. In our case an analytic continuation of the A(dS) pseudosphere is implied. The advantage of working on the compact space is that it admits a complete set of familiar basis functions, namely the spherical harmonics. The generating functional is,

$$Z(\eta,\bar{\eta},\chi_a) = \int \mathrm{d}\mu \exp(\int \mathrm{d}\Omega[\mathcal{L} + \bar{\eta}\widehat{\Psi} + \overline{\widehat{\Psi}}\eta + \chi\cdot\widehat{A}]) \tag{205}$$

where $\mathrm{d}\mu = [\mathrm{d}\overline{\widehat{\Psi}}][\mathrm{d}\widehat{\Psi}][\mathrm{d}\widehat{A}_a]$ is the functional measure including the Faddeev–Popov factor and $\mathrm{d}\Omega$ is the volume element.

Now the action (in four dimensions) is given by

$$\begin{aligned} S &= \int \mathrm{d}\Omega\left[\overline{\widehat{\Psi}}\frac{1}{l}(-sS\cdot L+2)\widehat{\Psi} + \mathrm{i}e\widehat{A}_a\hat{j}^a\right] \\ &= \int \mathrm{d}\Omega\left[\overline{\widehat{\Psi}}\frac{1}{l}(-sS\cdot L+2)\widehat{\Psi} - \mathrm{i}ep\widehat{A}_a\overline{\widehat{\Psi}}\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l}\widehat{\Psi}\right] \\ &= \int \mathrm{d}\Omega\overline{\widehat{\Psi}}\left[\frac{1}{l}(-sS\cdot L+2) - ig\widehat{A}_a\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l}\right]\widehat{\Psi} \\ &= \int \mathrm{d}\Omega\overline{\widehat{\Psi}}[Q - \mathrm{i}g\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l}\widehat{A}_a]\widehat{\Psi} \end{aligned} \tag{206}$$

where in the second line use has been made of (148) and $Q = \frac{1}{l}(-sS\cdot L+2)$, $g = ep$.

Here we will show the calculation for dS space (i.e. $s=-1$). AdS space calculation is similar to it. Following Fujikawa, let ϕ_n be a complete set of eigenfunction for the Dirac operator

$$D_A = Q - \mathrm{i}g\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l}\widehat{A}_a \tag{207}$$

i.e.

$$\begin{aligned} &D_A\phi_n = \lambda_n\phi_n \\ &\int \mathrm{d}\Omega\phi_n^\dagger(r)\phi_m(r) = \delta_{nm} \end{aligned} \tag{208}$$

Under the chirality transformation $\widehat{\Psi} \to \mathrm{e}^{\mathrm{i}\epsilon(r)\frac{\Gamma\cdot r}{l}}\widehat{\Psi}$, $\overline{\widehat{\Psi}} \to \overline{\widehat{\Psi}}\mathrm{e}^{\mathrm{i}\epsilon(r)\frac{\Gamma\cdot r}{l}}$ the functional measure transforms as $\mathrm{d}\mu \to \mathrm{d}\mu\exp[-2i\int \mathrm{d}\Omega\epsilon(r)A(r)]$, where

$$A(r) = \sum_n \phi_n^\dagger\frac{\Gamma\cdot r}{l}\phi_n \tag{209}$$

and the lagrangian transforms as $\mathcal{L} \to \mathcal{L} - \mathrm{i}\epsilon[L^{ab}(\overline{\widehat{\Psi}}S_{ab}\frac{\Gamma\cdot r}{l^2}\widehat{\Psi}) - 2\overline{\widehat{\Psi}}\frac{\Gamma\cdot r}{l^2}\widehat{\Psi}]$. Now the requirement of invariance of the generating functional under chiral transformation gives the correct anomaly equation, which turns out to be,

$$L^{ab}\left[\overline{\widehat{\Psi}}S_{ab}\frac{\Gamma\cdot r}{l^2}\widehat{\Psi}\right] - 2\overline{\widehat{\Psi}}\frac{\Gamma\cdot r}{l^2} = -2A(r) \tag{210}$$

Multiplying the above by r_c we obtain the final form of the anomaly equation as,

$$\hat{j}_{c5} - L_{cb}\hat{j}^{b5} = 2r_cA(r) \tag{211}$$

where $\hat{j}_{a5}$ is given by Eq. (159). $A(r)$ is the anomaly factor which will now be explicitly computed.

The conditionally convergent sum in $A(r)$ (209) is evaluated by regularizing large eigenvalues and changing to the free spinor harmonic basis. The spinor harmonics $\Psi^{(\mu)}_{l'ms}(r)$ are constructed to be orthonormalized eigenfunctions of the free massless Dirac operator $Q = \frac{1}{l}(S\cdot L + 2)$; i.e.

$$
\begin{aligned}
&Q\Psi^{(\mu)}_{l'ms} = \mu\Psi^{(\mu)}_{l'ms},; \quad \mu = -(l'+2)(l'+1),\\
&\Psi^{(\mu)}_{l'ms} = P^{(\mu)}Y_{l'm}\chi_s,\\
&P^{-(l'+2)} = \frac{l'+3-S\cdot L}{2l'+3},\\
&P^{(l'+1)} = \frac{1+S\cdot L}{2l'+3}
\end{aligned}
\tag{212}
$$

In the above set $Y_{l'm}(r)$ are four-dimensional spherical harmonics which satisfy,

$$
\begin{aligned}
&\int \mathrm{d}\Omega Y_{l'_1m_1}(r)Y_{l'_2m_2}(r) = \delta_{l'_1l'_2}\delta_{m_1m_2},\\
&\sum_m Y_{l'm}(r)Y_{l'm}(r') = \frac{2l'+3}{4\pi^{\frac{5}{2}}}\Gamma\left(\frac{3}{2}\right)C^{\frac{3}{2}}_{l'}\left(\frac{r\cdot r'}{l^2}\right),\\
&\sum_{l'm} Y_{l'm}(r)Y_{l'm}(r') = \delta(r-r')
\end{aligned}
\tag{213}
$$

where the index 'm' actually stands for the three 'magnetic' quantum numbers and the 'χ_s' are constant orthonormal 2^2 componant spinors. The $C^{(\frac{3}{2})}_{l'}$ are Gegenbauer polynomials.

Therefore,

$$
\begin{aligned}
A(r) &= \lim_{M\to\infty}\sum_n \phi^\dagger_n(r)\frac{\Gamma\cdot r}{l}\mathrm{e}^{-\left(\frac{\lambda_n}{M}\right)^2}\phi_n(r)\\
&= \lim_{M\to\infty}\sum_{l'\mu ms}\Psi^{(\mu)\,\dagger}_{l'ms}(r)\frac{\Gamma\cdot r}{l}\mathrm{e}^{-\left(\frac{D_A}{M}\right)^2}\Psi^{(\mu)}_{l'ms}(r)
\end{aligned}
\tag{214}
$$

Now from (207),

$$
D_A^2 = Q^2 - \mathrm{i}g(S\cdot L+2)\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l^2}\widehat{A}_a - \mathrm{i}g\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l^2}\widehat{A}_a(S\cdot L+2) - g^2\left(\theta^{ab}\Gamma_b\frac{\Gamma\cdot r}{l}\widehat{A}_a\right)^2
\tag{215}
$$

In the previous section we have seen that $\{S\cdot L+2, \frac{\Gamma\cdot r}{l}\} = 0$ (Eq. (197)). Also, one can show another anti-commuting relation,

$$
\left\{\theta^{ab}\Gamma_b, \frac{\Gamma\cdot r}{l}\right\} = 0.
\tag{216}
$$

Using these anti-commuting relations, (215) can be written as,

$$
\begin{aligned}
D_A^2 &= Q^2 - \mathrm{i}g\frac{\Gamma\cdot r}{l^2}(S\cdot L+2)\theta^{ab}\Gamma_b\widehat{A}_a + \mathrm{i}g\frac{\Gamma\cdot r}{l^2}\theta^{ab}\Gamma_b\widehat{A}_a(S\cdot L+2) + g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2\\
&= Q^2 - \mathrm{i}g\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a + g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2 + \mathrm{i}g\frac{\Gamma\cdot r}{l^2}\theta^{ab}\Gamma_b\widehat{A}_a S\cdot L
\end{aligned}
\tag{217}
$$

Also, after some simplifications one can show,

$$\Gamma \cdot r\theta^{ab}\Gamma_b\widehat{A}_a S\cdot L = \widehat{A}^a r_b \Gamma_a \Gamma_c L^{bc} \tag{218}$$

Therefore,

$$\begin{aligned} A(r) &= \lim_{M\to\infty}\sum_{l'\mu ms}\Psi^{(\mu)}_{l'ms}(r)\frac{\Gamma\cdot r}{l}\exp\left\{-M^{-2}\left[Q^2 - ig\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a\right.\right.\\ &\qquad \left.\left. + g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2 + \mathrm{i}g\frac{\Gamma\cdot r}{l^2}\theta^{ab}\Gamma_b\widehat{A}_a S.L\right]\right\}\Psi^{(\mu)}_{l'ms}(r)\\ &= \lim_{M\to\infty, r\to r'}\sum_{l'}\mathrm{Tr}\left[\frac{\Gamma\cdot r}{l}\exp\left\{-M^{-2}\left(Q^2 - \mathrm{i}g\frac{\Gamma\cdot r}{l^2}S.L\theta^{ab}\Gamma_b\widehat{A}_a\right.\right.\right.\\ &\qquad \left.\left.\left. + g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2 + \frac{\mathrm{i}g}{l^2}\widehat{A}^a r_b\Gamma_a\Gamma_c L^{bc}\right)\right\}\right] \times \frac{\Gamma\left(\frac{3}{2}\right)}{4\pi^{\frac{5}{2}}}(2l'+3)C^{\left(\frac{3}{2}\right)}_{l'}\left(\frac{r\cdot r'}{l^2}\right) \end{aligned} \tag{219}$$

Now one can check the following trace relation,

$$\mathrm{Tr}(\Gamma_a\Gamma_b\Gamma_c\Gamma_d\Gamma_e) = -4\mathrm{i}\epsilon_{abcde} \tag{220}$$

for four dimensions. The behaviour of the summand in (219) for large l' is $l'^3 \mathrm{e}^{-\frac{l'^2}{M^2}}$. Together with the above trace relation and (216) we can eliminate several terms in the exponentials. Then (219) simplifies to,

$$\begin{aligned} A(r) &= \lim_{M\to\infty, r\to r'}\sum_{l'}\mathrm{e}^{-\left(\frac{l'}{M}\right)^2}C^{\left(\frac{3}{2}\right)}_{l'}\left(\frac{r\cdot r'}{l^2}\right)\frac{\Gamma\left(\frac{3}{2}\right)}{4\pi^{\frac{5}{2}}}(2l'+3)\\ &\quad \times \mathrm{Tr}\left[\frac{\Gamma\cdot r}{l}\exp\left\{-\frac{1}{M^2}\left(-\mathrm{i}g\frac{\Gamma\cdot r}{l^2}S.L\theta^{ab}\Gamma_b\widehat{A}_a + g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2\right)\right\}\right]\\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{4\pi^{\frac{5}{2}}}\lim_{M\to\infty}\sum_{l'}\mathrm{e}^{-\left(\frac{l'}{M}\right)^2}(2l'+3)\frac{\Gamma(l'+3)}{\Gamma(l'+1)\Gamma(3)}\\ &\quad \times \mathrm{Tr}\left[\frac{\Gamma\cdot r}{l}\exp\left\{-\frac{1}{M^2}\left(-\mathrm{i}g\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a + g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2\right)\right\}\right] \end{aligned} \tag{221}$$

where we have used $C^b_a(1) = {}^{(a+2b-1)}C_a$ in the last line. The sum in (221) gives,

$$\begin{aligned} &\frac{\Gamma\left(\frac{3}{2}\right)}{2\pi^{\frac{5}{2}}\Gamma(3)}\sum_{l'}\mathrm{e}^{-\left(\frac{l'}{M}\right)^2}\left(l'+\frac{3}{2}\right)(l'+2)(l'+1)\\ &\quad = \frac{2}{(4\pi)^2}M\int \mathrm{d}l'\mathrm{e}^{-l'^2}[(Ml')^3 + O(Ml')^2] = \frac{1}{(4\pi)^2}M^4\left[1+O\left(\frac{1}{M}\right)\right] \end{aligned} \tag{222}$$

Expansion of the exponential in (221) gives terms like,

$$
\begin{aligned}
&\frac{1}{t!}\left(-\frac{1}{M^2}\right)^t \mathrm{Tr}\left[\frac{\Gamma\cdot r}{l}\left\{-\mathrm{i}g\frac{\Gamma\cdot r}{l^2}S.L\theta^{ab}\Gamma_b\widehat{A}_a+g^2(\theta^{ab}\Gamma_b\widehat{A}_a)^2\right\}^t\right]\\
&\quad=\frac{1}{t!}\left(-\frac{1}{M^2}\right)^t \mathrm{Tr}\left[\frac{\Gamma\cdot r}{l}\left\{-\mathrm{i}g\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a+g^2(\Gamma\cdot\widehat{A})^2\right\}^t\right]\\
&\quad=\frac{1}{t!}\left(\frac{\mathrm{i}g}{M^2}\right)^t \mathrm{Tr}\left[\frac{\Gamma\cdot r}{l}\left\{\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a+\mathrm{i}g(\Gamma\cdot\widehat{A})^2\right\}\right.\\
&\quad\left.\left\{\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a+\mathrm{i}g(\Gamma\cdot\widehat{A})^2\right\}^{t-1}\right]
\end{aligned}\tag{223}
$$

Using the value of θ^{ab} and S_{ab} the terms in the trace simplify to,

$$
\frac{\Gamma\cdot r}{l}\left\{\frac{\Gamma\cdot r}{l^2}S.L\theta^{ab}\Gamma_b\widehat{A}_a+\mathrm{i}g(\Gamma\cdot\widehat{A})^2\right\}=\frac{1}{2l}\Gamma_a\Gamma_b\Gamma_c(L^{ab}\widehat{A}^c-2\mathrm{i}gr^a\widehat{A}^b\widehat{A}^c)\tag{224}
$$

and

$$
\frac{\Gamma\cdot r}{l^2}S\cdot L\theta^{ab}\Gamma_b\widehat{A}_a+\mathrm{i}g(\Gamma\cdot\widehat{A})^2=-\frac{r_d}{l^2}\Gamma_e\Gamma_f(L^{de}\widehat{A}^f-\mathrm{i}gr^d\widehat{A}^e\widehat{A}^f)\tag{225}
$$

where for the last relation we have used,

$$
\begin{aligned}
r^a\Gamma_a\Gamma_b\Gamma_c\Gamma_dL^{bc}\widehat{A}^d&=r^a(-2\eta_{ab}-\Gamma_b\Gamma_a)\Gamma_c\Gamma_dL^{bc}\widehat{A}^d\\
&=-2r_b\Gamma_c\Gamma_dL^{bc}\widehat{A}^d-\Gamma_b\Gamma\cdot r\Gamma_c\Gamma_d(r^b\partial^c-r^c\partial^b)\widehat{A}^d\\
&=-2r_b\Gamma_c\Gamma_dL^{bc}\widehat{A}^d
\end{aligned}\tag{226}
$$

since $(\Gamma\cdot r)^2=l^2$. So, R.H.S of (223) becomes,

$$
\frac{1}{t!}(-1)^{t-1}\left(\frac{\mathrm{i}g}{M^2}\right)^t\mathrm{Tr}\left[\frac{1}{2l}\Gamma_a\Gamma_b\Gamma_c(L^{ab}\widehat{A}^c-2\mathrm{i}gr^a\widehat{A}^b\widehat{A}^c)\left\{\frac{r_d}{l^2}\Gamma_e\Gamma_f(L^{de}\widehat{A}^f-\mathrm{i}gr^d\widehat{A}^e\widehat{A}^f)\right\}^{t-1}\right]\tag{227}
$$

Now looking at (222) and (227) and considering the limit $M\to\infty$ it is seen that only the $t=2$ term contributes in (227). Therefore,

$$
\begin{aligned}
A(r)&=\frac{1}{4(4\pi)^2}g^2\mathrm{Tr}\left[\Gamma_a\Gamma_b\Gamma_c(L^{ab}\widehat{A}^c-2\mathrm{i}g\frac{r^a}{l}\widehat{A}^b\widehat{A}^c)\frac{r_d}{l^3}\Gamma_e\Gamma_f(L^{de}\widehat{A}^f-\mathrm{i}g\frac{r^d}{l}\widehat{A}^e\widehat{A}^f)\right]\\
&=-\mathrm{i}\frac{g^2}{16\pi^2l^3}\epsilon_{abcef}\left[r_d(L^{ab}\widehat{A}^c)(L^{de}\widehat{A}^f)+\mathrm{i}gl(L^{ab}\widehat{A}^c)\widehat{A}^e\widehat{A}^f\right.\\
&\quad\left.-2\mathrm{i}g\frac{r^a}{l}\widehat{A}^b\widehat{A}^cr_dL^{de}\widehat{A}^f+2r^a\widehat{A}^b\widehat{A}^c\widehat{A}^e\widehat{A}^f\right]\\
&=-\mathrm{i}\frac{g^2}{16\pi^2l^3}\epsilon_{abcef}r_d(L^{ab}\widehat{A}^c)(L^{de}\widehat{A}^f)
\end{aligned}\tag{228}
$$

where in the last line anti-symmetricity of ϵ_{abcef} has been used. Now from (45) we obtain (for abelian case),

$$
\widehat{F}^{abc}r_d\widehat{F}^{def}=6\epsilon_{abcef}L^{ab}\widehat{A}^cr_dL^{de}\widehat{A}^f+3\epsilon_{abcef}L^{ab}\widehat{A}^cr_dL^{ef}\widehat{A}^d\tag{229}
$$

Again, multiplying (48) from left by ϵ_{abcef} and from right by $r_d L_{ef} \widehat{A}_d$ we obtain,

$$\epsilon_{abcef}(L^{ab}K^{c\mu})r_d L^{ef}\widehat{A}^d = 0 \tag{230}$$

Therefore the last term in (229) reduces to,

$$\begin{aligned}\epsilon_{abcef}(L^{ab}\widehat{A}^c)r_d(L^{ef}\widehat{A}^d) &= \epsilon_{abcef}(L^{ab}K^{c\nu}A_\nu)r_d(L^{ef}\widehat{A}^d)\\ &= \epsilon_{abcef}(L^{ab}K^{c\nu})A_\nu r_d L^{ef}\widehat{A}^d + \epsilon_{abcef}K^{c\nu}(L^{ab}A_\nu)r_d L^{ef}\widehat{A}^d\\ &= \epsilon_{abcef}K^{c\nu}(L^{ab}A_\nu)r_d(L^{ef}K^{d\mu})A_\mu\\ &\quad + \epsilon_{abcef}K^{c\nu}(L^{ab}A_\nu)r_d K^{d\mu}(L^{ef}A_\mu) = 0\end{aligned} \tag{231}$$

where we have used (230) and $r_a K^{a\mu} = 0$. So (229) yields,

$$\widehat{F}^{abc} r_d \widehat{F}^{def} = 6\epsilon_{abcef} L^{ab}\widehat{A}^c r_d L^{de}\widehat{A}^f \tag{232}$$

Substituting this in (228) we obtain,

$$A(r) = -\mathrm{i}\frac{g^2}{96\pi^2 l^3}\epsilon_{abcef}\widehat{F}^{abc} r_d \widehat{F}^{def} = -\mathrm{i}\frac{g^2}{96\pi^2 l^3} r_a \epsilon_{bcdef}\widehat{F}^{abc}\widehat{F}^{def} \tag{233}$$

Putting everything in (211), we get the anomaly equation,

$$\hat{j}_{g5} - L_{gb}\hat{j}^{b5} = -\frac{\mathrm{i}g^2}{48\pi^2 l^3} r_g r_a \epsilon_{bcdef}\widehat{F}^{abc}\widehat{F}^{def}. \tag{234}$$

We now reproduce the above relation by stereographically projecting the familiar Adler-Bell-Jackiw anomaly [25] on the A(dS) pseudosphere. For the axial current, employing a gauge invariant regularisation, the familiar result on the flat space is known to be [25],

$$\partial_\mu j^{\mu 5} = \frac{\mathrm{i}g^2}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho} \tag{235}$$

Using (29) and the definition of the current (127) (appropriately interpreted for the axial vector currents), it is possible to obtain the identification,

$$r^a L_{ab}\hat{j}^{b5} = s l^2\left(1 + s\frac{x^2}{4l^2}\right)^4 \partial_\mu j^{\mu 5} \tag{236}$$

In getting at the final result, use was made of the identity (21). This provides a map for one side of (235). To obtain an analogous form for the other side, it is necessary to consider the completely anti-symmetric tensor $\epsilon_{\mu\nu\lambda\rho}$ whose value is the same in all systems.

In order to provide a mapping among the ϵ-tensors in the two spaces, we adopt the same rule (46) used for defining the antisymmetric field tensor. However there is a slight subtlety. Strictly speaking, this Levi–Civita epsilon is a tensor density. Hence its transformation law is modified by an appropriate conformal (weight) factor,

$$\epsilon_{abcde} = \frac{1}{l}\left(1 + s\frac{x^2}{4l^2}\right)^{-4}\left(r_a K^\mu_b K^\nu_c K^\lambda_d K^\rho_{\mathrm{e}} + \text{cyclic permutations in}(a,b,c,d,e)\right)\epsilon_{\mu\nu\lambda\rho} \tag{237}$$

It is possible to verify the above relation by an explicit calculation, taking the convention that both the epsilons are $+1(-1)$ for any even (odd) permutation of distinct entries (0, 1, 2, 3, 4) in that order.

The inverse relation is obtained from (237) by appropriate contractions and exploiting the identity (18),

$$\begin{aligned}\epsilon_{\mu\nu\lambda\rho} &= s\frac{1}{l}\left(1+s\frac{x^2}{4l^2}\right)^{-4} r_a K_{b\mu} K_{c\nu} K_{d\lambda} K_{e\rho} \epsilon^{abcde} \\ &= \frac{s}{l}\left(1+s\frac{x^2}{4l^2}\right)^{4} \frac{\partial r_b}{\partial x^\mu}\frac{\partial r_c}{\partial x^\nu}\frac{\partial r_d}{\partial x^\lambda}\frac{\partial r_e}{\partial x^\rho}(r_a \epsilon^{abcde})\end{aligned} \tag{238}$$

Now the explicit expressions for the anomaly are identified with the minimum of effort. Indeed, using (50) and (238), the ABJ anomaly is projected as,

$$\frac{1}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho} = s\frac{1}{16\pi^2 l^5}\left(1+s\frac{x^2}{4l^2}\right)^{-4} r^a r_f r_i \epsilon_{abcde}\widehat{F}^{fbc}\widehat{F}^{\text{i}de} \tag{239}$$

The weight factors cancel out from both sides of the projected anomaly Eq. (235) and we obtain, using (236) and (239),

$$r_a L^{ab}\hat{j}_{b5} = \frac{\text{i}g^2}{16\pi^2 l^3} r^a r_f r_i \epsilon_{abcde}\widehat{F}^{fbc}\widehat{F}^{\text{i}de} \tag{240}$$

It is also possible to rewrite the anomaly expression in a form that resembles the expression on the hypersphere [18,27]. To do this we have to exploit the identity,

$$\epsilon_{abcde}\widehat{F}^{abc} = s\frac{3}{l^2} r^a r_f \epsilon_{abcde}\widehat{F}^{fbc} \tag{241}$$

Then the A(dS) anomaly equation reduces to,

$$r_a L^{ab}\hat{j}_{b5} = s\frac{\text{i}g^2}{48\pi^2 l} r_a \epsilon_{bcdef}\widehat{F}^{abc}\widehat{F}^{def} \tag{242}$$

This is the desired anomalous current divergence equation in the A(dS) space which has a close resemblance with the corresponding equation on the hypersphere. It is the exact analogue of the ABJ-anomaly equation on the flat space.

There is another way in which the anomaly equation can be expressed. To see this, observe that the projection (158) for the current corresponds to $n = 2$ in the general formula (87). Hence the identity (88) holds and we obtain,

$$r_c r_a L^{ab}\hat{j}_{b5} = s l^2 (L_{cb}\hat{j}^{b5} - \hat{j}_{c5}) \tag{243}$$

Thus the anomaly Eq. (242) takes the form given in (234), thereby completing our proof of equivalence. Compatibility between the two forms (242) and (234) is easily established by contracting the latter with r^g and using the transversality of the current $(r^a\hat{j}_{a5} = 0)$.

The normal Ward identity for the vector current is obtained by setting the right hand side of either (242) or (234) equal to zero.

It is known that on the flat space, it is feasible to redefine the current so that the anomaly vanishes. In that case, however, the current is no longer gauge invariant. This compensating term is given by

$$X^\mu = \frac{ig^2}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}A_\nu F_{\lambda\rho} \tag{244}$$

such that $\partial_\mu J^{\mu 5} = 0$, where,

$$J^{\mu 5} = j^{\mu 5} - X^\mu. \tag{245}$$

Observe that $J^{\mu 5}$ is not gauge invariant due to the presence of X^μ.

The same phenomenon also occurs on the A(dS) space. Here the compensating piece is obtained from a projection of (244),

$$\widehat{X}_a = \left(1 + s\frac{x^2}{4l^2}\right)^2 K_a^\mu X_\mu \tag{246}$$

Since,

$$r_a L^{ab}\widehat{X}_b = sl^2\left(1 + s\frac{x^2}{4l^2}\right)^4 \partial_\mu X^\mu = s\frac{ig^2}{48\pi^2 l} r_a \epsilon_{bcdef}\widehat{F}^{abc}\widehat{F}^{def} \tag{247}$$

we observe that the modified current,

$$\widehat{J}_{a5} = \hat{j}_{a5} - \widehat{X}_a = \left(1 + s\frac{x^2}{4l^2}\right)^2 K_a^\mu J_{\mu 5} \tag{248}$$

is anomaly free, i.e. $r_a L_{ab}\widehat{J}_{b5} = 0$. Also note that the anomaly free currents are mapped in the same way as the anomalous ones. The transversality condition $r^a\widehat{J}_{a5} = 0$ is obviously satisfied by (248). Expectedly, the current $\widehat{J}_{a5}$ is not gauge invariant.

It is straightforward to extend this calculation for arbitrary $D = 2n$ dimensions.[4] The flat space expression (235) is known to be generalised as [28,29],

$$\partial_\mu j^{\mu 5} = \frac{2i}{(4\pi)^n n!}\epsilon_{\mu_1\mu_2\ldots\mu_{2n}} F^{\mu_1\mu_2}F^{\mu_3\mu_4}\ldots F^{\mu_{2n-1}\mu_{2n}} \tag{249}$$

The map for the Levi–Civita tensor is the generalised version of (238),

$$\epsilon_{\mu_1\mu_2\ldots\mu_{2n}} = \frac{s}{l}\left(1 + s\frac{x^2}{4l^2}\right)^{-2n} r_a K_{a_1\mu_1}K_{a_2\mu_2}\ldots K_{a_{2n}\mu_{2n}}\epsilon^{aa_1a_2\ldots a_{2n}} \tag{250}$$

while the map for the field tensor is given by (50). Using these mappings the projected expression for the anomaly is,

$$\begin{aligned}\frac{2}{(4\pi)^n n!}\epsilon_{\mu_1\mu_2\ldots\mu_{2n}}F^{\mu_1\mu_2}\ldots F^{\mu_{2n-1}\mu_{2n}} &= \frac{2}{(4\pi)^n n!}\frac{s}{l^{2n+1}}\left(1 + s\frac{x^2}{4l^2}\right)^{-2n} \\ &\times r^a r_{a_1} r_{a_2}\ldots r_{a_n}\epsilon_{ab_1b_2\ldots b_{2n}}\widehat{F}^{a_1b_1b_2}\widehat{F}^{a_2b_3b_4}\ldots\widehat{F}^{a_nb_{2n-1}b_{2n}}\end{aligned} \tag{251}$$

Next, the projection of the L.H.S. of (235) has to be found. Using (29) and (135) we obtain,

[4] For simplicity, the coupling factor g is not included.

$$r_a L^{ab}\hat{j}_b^5 = sl^2\left(1 + s\frac{x^2}{4l^2}\right)^{2n}\partial^\mu j_\mu^5 \tag{252}$$

Finally, exploiting (251) and (252) the projected form of (249) is derived,

$$r_a L^{ab}\hat{j}_b^5 = \frac{2\mathrm{i}}{(4\pi)^n n!}\frac{1}{l^{2n-1}} r^a r_{a_1} r_{a_2}\ldots r_{a_n}\epsilon_{ab_1b_2\ldots b_{2n}}\widehat{F}^{a_1b_1b_2}\widehat{F}^{a_2b_3b_4}\ldots\widehat{F}^{a_nb_{2n-1}b_{2n}} \tag{253}$$

Following the same steps employed for obtaining (234) from (240), the above anomaly equation is expressed as,

$$\hat{j}_{g5} - L_{gb}\hat{j}^{b5} = -\frac{2\mathrm{i}}{3(4\pi)^n n! l^{2n-1}} r_g r_a \epsilon_{a_1a_2\ldots a_{2n+1}}\widehat{F}^{aa_1a_2}\widehat{F}^{a_3a_4a_5}\ldots\widehat{F}^{a_{2n-1}a_{2n}a_{2n+1}} \tag{254}$$

5.2. Non-abelian chiral anomalies

Now we will discuss about the non-abelian chiral anomaly of spin one-half fermions. As is well known [28–31], there are two types of anomaly on the flat Minkowski space: the covariant anomaly and the consistent anomaly. The covariant anomaly, as its name implies, transforms covariantly under the gauge transformation. The consistent anomaly, on the other hand, is the one that satisfies the Wess-Zumino consistency condition. Just as the covariant anomaly does not satisfy this condition, the consistent anomaly does not transform covariantly.

The covariant anomaly is given by

$$(D_\mu j^\mu)^{(\alpha)}_{(\text{covariant})} = (\partial_\mu j^\mu)^{(\alpha)} - \mathrm{i}[A_\mu, j^\mu]^{(\alpha)} = \frac{\mathrm{i}}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\cdot\{\lambda^\alpha F_{\mu\nu}F_{\rho\sigma}\} \tag{255}$$

where λ^α are the symmetry matrices and $(j^\mu)^{(\alpha)}$ is the chiral current,

$$(j_\mu)^{(\alpha)} = \overline{\Psi}\lambda^\alpha\gamma_\mu\frac{1+\gamma_5}{2}\Psi \tag{256}$$

The result (255) has been obtained by various methods [28–31], all of which basically rely on regularising the current (256) in a covariant manner.

We shall now stereographically project (255) to obtain the covariant anomaly on the A(dS) space. The map for the current (256) is similar to (127) and is given by

$$(\hat{j}_a)^{(\alpha)} = \left(1 + s\frac{x^2}{4l^2}\right)^2 K_a^\mu (j_\mu)^{(\alpha)} \tag{257}$$

Now, using (39) and following steps similar to the abelian case, we find,

$$(r^a\widehat{\mathcal{L}}_{ab}\hat{j}^b)^{(\alpha)} = sl^2\left(1 + s\frac{x^2}{4l^2}\right)^4 (D_\mu j^\mu)^{(\alpha)} \tag{258}$$

which is the non-abelian version of (236).

The R.H.S. of (255) is obtained by a straightforward generalisation of (239),

$$\frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\cdot\{\lambda^\alpha F_{\mu\nu}F_{\rho\sigma}\} = \frac{s}{32\pi^2 l^5}\left(1 + s\frac{x^2}{4l^2}\right)^{-4} r^a r_f r_i \epsilon_{abcde}\mathrm{Tr}\cdot\{\lambda^\alpha\widehat{F}^{fbc}\widehat{F}^{ide}\} \tag{259}$$

Hence, the covariant anomaly on A(dS) space is given by

$$(r^f \widehat{\mathcal{L}}_{fg} \hat{j}^g)^{(\alpha)}_{(\text{covariant})} = \frac{\mathrm{i}s}{96\pi^2 l} r_a \epsilon_{bcdef} \mathrm{Tr} \cdot \{\lambda^\alpha \widehat{F}^{abc} \widehat{F}^{def}\} \tag{260}$$

which follows on exploiting (255), (258), (259) and the identity (241).

Next, the consistent anomaly is considered. On the flat Minkowski space this is given by

$$(D_\mu j^\mu)^{(\alpha)}_{(\text{consistent})} = \frac{\mathrm{i}}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \cdot \left\{ \lambda^\alpha \partial_\mu (A_\nu \partial_\rho A_\sigma - \frac{\mathrm{i}}{2} A_\nu A_\rho A_\sigma) \right\} \tag{261}$$

In order to project this equation it is convenient to recast it in the following form,

$$\begin{aligned} (D_\mu j^\mu)_{(\text{consistent})} &= \mathcal{A}^{(\alpha)}_{(\text{consistent})} \\ &= \frac{\mathrm{i}}{96\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \cdot \{\lambda^\alpha (F_{\mu\nu} F_{\rho\sigma} + iF_{\mu\nu} A_\rho A_\sigma + iA_\mu A_\nu F_{\rho\sigma} - \mathrm{i} A_\nu F_{\mu\rho} A_\sigma)\} \end{aligned} \tag{262}$$

where the definition (34) of field tensor $F_{\mu\nu}$ has been used. Using the inverse maps for $\epsilon^{\mu\nu\rho\sigma}$ (238), field tensor (50) and the vector field (22), the stereographic projection of the consistent anomaly (262) on A(dS) space is,

$$\begin{aligned} (\mathcal{A})^{(\alpha)}_{(\text{consistent})} = \frac{\mathrm{i}}{96\pi^2 l^3} \left(1 + s\frac{x^2}{4l^2}\right)^{-4} \epsilon_{abcde} r^a r_f \left[\frac{s}{l^2} r_i \mathrm{Tr} \cdot \{\lambda^\alpha \widehat{F}^{fbc} \widehat{F}^{ide}\} \right. \\ \left. + \mathrm{i}\mathrm{Tr} \cdot \{\lambda^\alpha (\widehat{F}^{fbc} \widehat{A}^d \widehat{A}^e + \widehat{A}^c \widehat{A}^d \widehat{F}^{fbc} - \widehat{A}^c \widehat{F}^{fbd} \widehat{A}^e)\} \right] \end{aligned} \tag{263}$$

Now using the identity (241) one can show,

$$\begin{aligned} (\mathcal{A})^{(\alpha)}_{(\text{consistent})} = \frac{\mathrm{i}}{288\pi^2 l} (1 + s\frac{x^2}{4l^2})^{-4} \left[\frac{1}{l^2} r_a \epsilon_{bcdef} \mathrm{Tr} \cdot \{\lambda^\alpha \widehat{F}^{abc} \widehat{F}^{def}\} \right. \\ \left. + \mathrm{i}s \epsilon_{abcde} \mathrm{Tr} \cdot \{\lambda^\alpha (\widehat{F}^{abc} \widehat{A}^d \widehat{A}^e + \widehat{A}^c \widehat{A}^d \widehat{F}^{abc} - \widehat{A}^c \widehat{F}^{abd} \widehat{A}^e)\} \right] \end{aligned} \tag{264}$$

Therefore, the final equation for the consistent anomaly on the A(dS) space is,

$$\begin{aligned} (r^f \widehat{\mathcal{L}}_{fg} \hat{j}^b)^{(\alpha)}_{(\text{consistent})} = s\frac{\mathrm{i}l}{288\pi^2} \left[\frac{1}{l^2} r_a \epsilon_{bcdef} \mathrm{Tr} \cdot \{\lambda^\alpha \widehat{F}^{abc} \widehat{F}^{def}\} \right. \\ \left. + \mathrm{i}s \epsilon_{abcde} \mathrm{Tr} \cdot \{\lambda^\alpha (\widehat{F}^{abc} \widehat{A}^d \widehat{A}^e + \widehat{A}^d \widehat{A}^e \widehat{F}^{abc} + \widehat{A}^d \widehat{F}^{abc} \widehat{A}^e)\} \right] \end{aligned} \tag{265}$$

It is known that, on the flat Minkowski space, the covariant and consistent currents are related by a local counterterm given by

$$\begin{aligned} (X^\mu)^{(\alpha)} &= (j^\mu)^{(\alpha)}_{(\text{consistent})} - (j^\mu)^{(\alpha)}_{(\text{covariant})} \\ &= -\frac{\mathrm{i}}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \cdot \{\lambda^\alpha (A_\nu F_{\rho\sigma} + F_{\rho\sigma} A_\nu + iA_\nu A_\rho A_\sigma)\} \end{aligned} \tag{266}$$

As was shown in [29], this ambiguity in the current is such that it does not affect the effective action. This is a consequence of the fact that,

$$(A^\mu)^{(\alpha)} (X_\mu)^{(\alpha)} = 0 \tag{267}$$

From (266) we observe that the covariant and consistent anomalies on flat space are related by

$$(D_\mu j^\mu)^{(\alpha)}_{(\text{covariant})} = (D_\mu j^\mu)^{(\alpha)}_{(\text{consistent})} - (D_\mu X^\mu)^{(\alpha)} \tag{268}$$

The above analysis is now carried out on the A(dS) space. The projection of (266) yields,

$$(\widehat{X}_b)^{(\alpha)} = \left(1 + s\frac{x^2}{4l^2}\right)^2 K_{b\mu}(X^\mu)^{(\alpha)}$$
$$= -\frac{\mathrm{i}}{48\pi^2 l} r^a \epsilon_{abcde}\left[\frac{1}{l^2} r_i \mathrm{Tr}\cdot\{\lambda^\alpha(\widehat{A}^c\widehat{F}^{ide} + \widehat{F}^{ide}\widehat{A}^c)\} + \mathrm{i}s\mathrm{Tr}\cdot\{\lambda^\alpha\widehat{A}^c\widehat{A}^d\widehat{A}^e\}\right] \quad (269)$$

Note that the condition analogous to (267) is,

$$(\widehat{A}^a)^{(\alpha)}(\widehat{X}_a)^{(\alpha)} = 0 \quad (270)$$

which is obviously satisfied by (269). Now, using the definition (39) of 'covariantised angular momentum' and (269), we have,

$$(r^f\widehat{\mathcal{L}}_{fg}\widehat{X}^g)^{(\alpha)} = -s\frac{\mathrm{i}l}{288\pi^2}\left[\frac{2}{l^2} r_a \epsilon_{bcdef}\mathrm{Tr}\cdot\{\lambda^\alpha\widehat{F}^{abc}\widehat{F}^{def}\} - \mathrm{i}s\epsilon_{abcde}\mathrm{Tr}\right.$$
$$\left.\cdot\{\lambda^\alpha(\widehat{F}^{abc}\widehat{A}^d\widehat{A}^e + \widehat{A}^d\widehat{A}^e\widehat{F}^{abc} + \widehat{A}^d\widehat{F}^{abc}\widehat{A}^e)\}\right] \quad (271)$$

Exploiting the expressions for the consistent anomaly, covariant anomaly and the above relation we have,

$$(r^f\widehat{\mathcal{L}}_{fg}\hat{j}^g)^{(\alpha)}_{(\mathrm{covariant})} = (r^f\widehat{\mathcal{L}}_{fg}\hat{j}^g)^{(\alpha)}_{(\mathrm{consistent})} - (r^f\widehat{\mathcal{L}}_{fg}\widehat{X}^g)^{(\alpha)} \quad (272)$$

which is the A(dS) space analogue of (268). Here, $(r^f\widehat{\mathcal{L}}_{fg}\widehat{X}^g)^{(\alpha)}$ plays the role of the local counterterm for the anomaly on the A(dS) space.

6. Duality symmetry

The well known electric-magnetic duality symmetry swapping field equations with the Bianchi identity in flat space has an exact counterpart on the A(dS) hyperboloid. To see this it is essential to introduce the dual field tensor that enters the Bianchi identity. The dual tensor is defined by

$$\tilde{F}_{ab} = -\frac{1}{6}\epsilon_{abcde}\widehat{F}^{cde} \quad (273)$$

Using (46) and (237) together with the properties of the Killing vectors the dual on the A(dS) space is expressed in terms of the dual on the flat space as,

$$\tilde{F}_{ab} = -slK_a^\lambda K_b^\rho\tilde{F}_{\lambda\rho} \quad (274)$$

where the flat space dual is given by

$$\tilde{F}_{\lambda\rho} = \frac{1}{2}\epsilon_{\lambda\rho\mu\nu}F^{\mu\nu} \quad (275)$$

Inversion of (274) yields,

$$\tilde{F}_{\mu\sigma} = -s\frac{1}{l}\left(1 + s\frac{x^2}{4l^2}\right)^{-4} K_{a\mu}K_{b\sigma}\tilde{F}ab = -s\frac{1}{l}\frac{\partial r_a}{\partial x^\mu}\frac{\partial r_b}{\partial x^\sigma}\tilde{F}^{ab} \quad (276)$$

where we have used (14) to obtain the final result. Apart from a dimensional scale the mapping is exactly identical to that of a second rank tensor given in (51).

The Bianchi identity on the A(dS) space is then given by

$$r_a L^{ab}\tilde{F}_{bc} = 0 \tag{277}$$

This is confirmed by a direct calculation. Alternatively, it becomes transparent by projecting it on the flat space by means of Killing vectors. Using the basic definitions and the identity,

$$K_b^\rho \partial_\rho (K^{b\mu} K_c^\nu)\tilde{F}_{\mu\nu} = 0 \tag{278}$$

we obtain,

$$r_a L^{ab}\tilde{F}_{bc} = -l^3 K^{b\mu} K_b^\lambda K_c^\rho \partial_\mu \tilde{F}_{\lambda\rho} \tag{279}$$

Finally, exploiting (18) we get the desired projection,

$$r_a L^{ab}\tilde{F}_{bc} = -l^3\left(1+s\frac{x^2}{4l^2}\right)^2 K_c^\rho \partial^\lambda \tilde{F}_{\lambda\rho} \tag{280}$$

which vanishes since $\partial^\lambda \tilde{F}_{\lambda\rho} = 0$.

Now the abelian equation of motion following from a variation of the action (72) is given by

$$L_{ab}\widehat{F}^{abc} = 0 \tag{281}$$

The duality transformation is next discussed. Analogous to the flat space rule, $\tilde{F} \to F; F \to -\tilde{F}$ the duality map here is provided by

$$\tilde{F}_{ab} \to \frac{r^c}{l}\widehat{F}_{abc}; \quad \frac{r^c}{l}\widehat{F}_{abc} \to -\tilde{F}_{ab} \tag{282}$$

It is easy to check the consistency of this map. The inverse of (273) yields,

$$\widehat{F}_{abc} = s\frac{1}{2}\epsilon_{abcde}\tilde{F}^{de} \tag{283}$$

while,

$$\widehat{F}^{abc} = -\frac{1}{2}\epsilon^{abcde}\tilde{F}_{de} \tag{284}$$

Under the first of the maps in (282), the above relation is transformed as,

$$\widehat{F}_{abc} \to -s\frac{1}{4l} r_f \epsilon_{abcde}\epsilon^{defgh}\tilde{F}_{gh} = -s\frac{1}{l}\left(r_a\tilde{F}_{bc} + r_b\tilde{F}_{ca} + r_c\tilde{F}_{ab}\right) \tag{285}$$

where use was made of (284) at an intermediate step. Contracting the above map by r^c immediately leads to the second relation in (282).

Likewise, under the map (285), the dual field (273) transforms as,

$$\tilde{F}_{ab} \to s\frac{1}{2l}\epsilon_{abcde} r^c \tilde{F}^{de} \tag{286}$$

Substituting the expression (273) for $\tilde{F}^{de}$ we reproduce the first of the maps given in (282).

Now the effect of the duality map on the equation of motion (281) is considered. Using (285) and the correspondence (274) along with the identity (278) we find,

$$\frac{1}{2}L_{ab}\widehat{F}^{abc} \to sl^2\left(1+s\frac{x^2}{4l^2}\right)^2 K^{c\rho}\partial^\mu \tilde{F}_{\mu\rho} \tag{287}$$

Finally, using (280) we obtain the cherished mapping,

$$\frac{1}{2}L_{ab}\widehat{F}^{abc} \rightarrow -s\frac{1}{l}r_aL^{ab}\tilde{F}_{bc} \tag{288}$$

showing how the equation of motion passes over to the Bianchi identity. Likewise the other map swaps the Bianchi identity to the equation of motion,

$$s\frac{1}{l}r_aL^{ab}\tilde{F}_{bc} \rightarrow \frac{1}{2}L_{ab}\widehat{F}^{abc} \tag{289}$$

It is feasible to perform a continuous $SO(2)$ duality rotations through an angle θ. The relevant transformations are then given by,

$$\frac{r^c}{l}\widehat{F}'_{abc} = \cos\theta\frac{r^c}{l}\widehat{F}_{abc} - \sin\theta\tilde{F}_{ab} \tag{290}$$

$$\tilde{F}'_{ab} = \sin\theta\frac{r^c}{l}\widehat{F}_{abc} + \cos\theta\tilde{F}_{ab} \tag{291}$$

This mixes the equation of motion and the Bianchi identity in the following way,

$$\frac{1}{2}L^{ab}\widehat{F}'_{abc} = \cos\theta\frac{1}{2}L^{ab}\widehat{F}_{abc} - \sin\theta s\frac{1}{l}r_aL^{ab}\tilde{F}_{bc} \tag{292}$$

$$s\frac{1}{l}r_aL^{ab}\tilde{F}'_{bc} = \sin\theta\frac{1}{2}L^{ab}\widehat{F}_{abc} + \cos\theta s\frac{1}{l}r_aL^{ab}\tilde{F}_{bc} \tag{293}$$

The discrete duality transformation corresponds to $\theta = \frac{\pi}{2}$.

7. Formulation of anti-symmetric tensor gauge theory

Our analysis can be extended to include higher rank tensor gauge theories. Some typical examples are the linearised version of gravity which uses a symmetric second rank tensor or the p-form gauge theories employing anti-symmetric tensor fields.

In this section we discuss our formulation for the second rank antisymmetric tensor gauge theory. Also, there are some features which distinguish it from the analysis for the vector gauge theory. The extension for higher forms is obvious. Both abelian and non-abelian theories will be considered. To set up the formulation it is convenient to begin with the abelian case which can be subsequently generalised to the non-abelian version. The action for a free 2-form gauge theory in flat four-dimensional Minkowski space is given by [32],

$$S = -\frac{1}{12}\int d^4xF^{\mu\nu\rho}F_{\mu\nu\rho} \tag{294}$$

where the field strength is defined in terms of the basic field as,

$$F_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{295}$$

The infinitesimal gauge symmetry is given by the transformation,

$$\delta B_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu \tag{296}$$

which is reducible since it trivialises for the choice $\Lambda_\mu = \partial_\mu\lambda$.

It is sometimes useful to express the action (or the lagrangian) in a first order form by introducing an extra field,

$$\mathcal{L} = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}B^{\rho\sigma} + \frac{1}{8}A^{\mu}A_{\mu} \tag{297}$$

where the $B \wedge F$ term involves the field tensor,

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{298}$$

Eliminating the auxiliary A_{μ} field by using its equation of motion, the previous form (294) is reproduced. The gauge symmetry is given by (296) together with $\delta A_{\mu} = 0$. The first order form is ideal for analysing the non-abelian theory.

To express the theory on the A(dS) pseudosphere, the mapping of the tensor field is first given. From the previous analysis, it is simply given by

$$\widehat{B}_{ab} = K_a^{\mu}K_b^{\nu}B_{\mu\nu} \tag{299}$$

and satisfies the transversality condition,

$$r^a\widehat{B}_{ab} = r^b\widehat{B}_{ab} = 0. \tag{300}$$

The tensor field with the latin indices is defined on the pseudosphere while those with the greek symbols are the usual one on the flat space. This is written in component notation by using the explicit form for the Killing vectors given in (12) and (13),

$$\widehat{B}_{\mu\nu} = \left(1 + s\frac{x^2}{4l^2}\right)\left(\left(1 + s\frac{x^2}{4l^2}\right)B_{\mu\nu} - s\frac{x^{\rho}x_{\nu}}{2l^2}B_{\mu\rho} - s\frac{x^{\rho}x_{\mu}}{2l^2}B_{\rho\nu}\right) \tag{301}$$

and,

$$\widehat{B}_{\mu 4} = \frac{1}{l}\left(1 + s\frac{x^2}{4l^2}\right)x^{\rho}B_{\mu\rho} \tag{302}$$

These are the analogues of (16). The inverse relation is given by,

$$\left(1 + s\frac{x^2}{4l^2}\right)^4 B^{\mu\nu} = K_a^{\mu}K_b^{\nu}\widehat{B}^{ab} \tag{303}$$

which may also be put in the form,

$$\left(1 + s\frac{x^2}{4l^2}\right)^2 B_{\mu\nu} = \widehat{B}_{\mu\nu} - s\frac{x_{\mu}\widehat{B}_{\nu 4}}{2l} + s\frac{x_{\nu}\widehat{B}_{\mu 4}}{2l} \tag{304}$$

which is the direct analogue of (17). Moreover using (14), the map (303) may also be expressed in a more transparant form as,

$$B_{\mu\nu} = \frac{\partial r_a}{\partial x^{\mu}}\frac{\partial r_b}{\partial x^{\nu}}\widehat{B}^{ab} \tag{305}$$

which is just the result (51) for a second rank tensor.

Next, the gauge transformations are discussed. From (296), the defining relation (299) and the angular momentum operator (28), infinitesimal transformations are given by

$$\delta\widehat{B}_{ab} = s\frac{r^c}{l^2}\left(K_b^{\mu}L_{ca} - K_a^{\mu}L_{cb}\right)\Lambda_{\mu} \tag{306}$$

This is consistent with $r^a\delta\widehat{B}_{ab} = 0$ which is imposed by (300). In this form the expression is not manifestly covariant. This may be contrasted with (38) which has this desirable feature. The point is that an appropriate map of the gauge parameter is necessary. In the previous example the gauge parameter was a scalar which retained its form. Here, since it is a vector, the required map is provided by a relation like (8), so that,

$$\widehat{\Lambda}_a = K_a^\mu \Lambda_\mu \tag{307}$$

Pushing the Killing vectors through the angular momentum operator and using the above map yields, after some simplifications,

$$\delta\widehat{B}_{ab} = s\frac{1}{l^2}\left[r^c\left(L_{ca}\widehat{\Lambda}_b - L_{cb}\widehat{\Lambda}_a\right) - r_a\widehat{\Lambda}_b + r_b\widehat{\Lambda}_a\right] \tag{308}$$

It is also reassuring to note that (308) manifests the reducibility of the gauge transformations. Since $\Lambda_\mu = \partial_\mu\lambda$ leads to a trivial gauge transformation in flat space, it follows from (307) that the corresponding feature should be present in the pseudospherical formulation when,

$$\widehat{\Lambda}_a = s\frac{1}{l^2}r^c L_{ca}\lambda \tag{309}$$

It is easy to check that with this choice, the gauge transformation (308) trivialises; i.e. $\delta\widehat{B}_{ab} = 0$.

The field tensor on the pseudosphere is constructed from the usual one given in (295). Since the Killing vectors play the role of the metric in connecting the two surfaces, this expression is given by a natural extension of (46),

$$\widehat{F}_{abcd} = \left(r_a K_b^\mu K_c^\nu K_d^\rho + r_b K_c^\mu K_a^\nu K_d^\rho + r_c K_d^\mu K_a^\nu K_b^\rho + r_d K_a^\mu K_c^\nu K_b^\rho\right) F_{\mu\nu\rho} \tag{310}$$

Note that cyclic permutations have to taken carefully since there is an even number of indices. The inverse mapping is provided by

$$F_{\mu\nu\rho} = s\frac{1}{l^2}\left(1 + s\frac{x^2}{4l^2}\right)^{-6} K_{b\mu}K_{c\nu}K_{d\rho}\left(r_a\widehat{F}^{abcd}\right) = s\frac{1}{l^2}\frac{\partial r_b}{\partial x^\mu}\frac{\partial r_c}{\partial x^\nu}\frac{\partial r_d}{\partial x^\rho}\left(r_a\widehat{F}^{abcd}\right) \tag{311}$$

This is a particular case of the general result (52).

In terms of the basic variables, the field tensor is known to be expressed as,

$$\widehat{F}_{abcd} = \left(L_{ab}\widehat{B}_{cd} + L_{bc}\widehat{B}_{ad} + L_{bd}\widehat{B}_{ca} + L_{ca}\widehat{B}_{bd} + L_{da}\widehat{B}_{cb} + L_{cd}\widehat{B}_{ab}\right) \tag{312}$$

To show that (310) is equivalent to (312), the same strategy as before, is adopted. Using the definition of the angular momentum (28), (312) is simplified as,

$$\widehat{F}_{abcd} = \left(r_a K_b^\mu - r_b K_a^\mu\right)\partial_\mu\left(K_c^\nu K_d^\sigma B_{\nu\sigma}\right) + \cdots \tag{313}$$

where the carets denote the inclusion of other similar (cyclically permuted) terms. Now there are two types of contributions. Those where the derivatives act on the Killing vectors and those where they act on the fields. The first class of terms cancel out as a consequence of an identity that is an extension of (48). The other class combines to reproduce (310).

The action on the A(dS) pseudosphere is now obtained by first taking a repeated product of the field tensor (310). Using the properties of the Killing vectors, this yields,

$$\widehat{F}_{abcd}\widehat{F}^{abcd} = 4sl^2\left(1+s\frac{x^2}{4l^2}\right)^6 F_{\mu\nu\rho}F^{\mu\nu\rho} \tag{314}$$

From the definition of the flat space action (294) and the volume element (23), it follows that the above identification leads to the pseudospherical action,

$$S_\Omega = -s\frac{1}{48l^2}\int \mathrm{d}\Omega\left(1+s\frac{x^2}{4l^2}\right)^{-2}\widehat{F}_{abcd}\widehat{F}^{abcd} \tag{315}$$

Thus, up to a conformal factor, the corresponding lagrangian is given by,

$$\mathcal{L}_\Omega = -s\frac{1}{48l^2}\widehat{F}_{abcd}\widehat{F}^{abcd} \tag{316}$$

By its very construction this lagrangian would be invariant under the gauge transformation (308). There is however another type of gauge symmetry which does not seem to have any analogue in the flat space. To envisage such a possibility, consider a transformation of the type,[5]

$$\delta\widehat{B}_{ab} = L_{ab}\lambda \tag{317}$$

which could be a meaningful gauge symmetry operation on the A(dS) space. However, in flat space, it leads to a trivial gauge transformation. To see this explicitly, consider the effect of (317) on (303),

$$\left(1+s\frac{x^2}{4l^2}\right)^4\delta B^{\mu\nu} = K^\mu_a K^\nu_b L_{ab}\lambda \tag{318}$$

Inserting the expression for the angular momentum from (28) and exploiting the transversality (6) of the Killing vectors, it follows that,

$$\delta B_{\mu\nu} = 0 \tag{319}$$

thereby proving the statement. To reveal that (317) indeed leaves the lagrangian (316) invariant, it is desirable to recast it in the form,

$$\mathcal{L}_\Omega = -s\frac{1}{32l^2}\widehat{\Sigma}_a\widehat{\Sigma}^a \tag{320}$$

where

$$\widehat{\Sigma}_a = \epsilon_{abcde}L^{bc}\widehat{B}^{de} \tag{321}$$

Under the gauge transformation (317), a simple algebra shows that $\delta\widehat{\Sigma}_a = 0$ and hence the lagrangian remains invariant.

The inclusion of a non-abelian gauge group is feasible. Results follow logically from the abelian theory with suitable insertion of the non-abelian indices. As remarked earlier it is useful to consider the first order form (297). The lagrangian is given by its straightforward generalisation, where the non-abelian field strength has already been defined in (34). It is gauge invariant under the non-abelian generalisation of (296) with the ordinary derivatives replaced by the covariant derivatives with respect to the potential A_μ, and $\delta A_\mu = 0$. By the

[5] Recently such a transformation was considered on the hypersphere [33,18].

help of our equations it is possible to project this lagrangian on the A(dS) space. For instance, the corresponding gauge transformations look like,

$$\delta\widehat{B}_{ab} = s\frac{1}{l^2}\left[r^c\left(L_{ca}\widehat{\Lambda}_b - L_{cb}\widehat{\Lambda}_a\right) - r_a\widehat{\Lambda}_b + r_b\widehat{\Lambda}_a\right] - \mathrm{i}[\widehat{A}_a, \widehat{\Lambda}_b] + \mathrm{i}[\widehat{A}_b, \widehat{\Lambda}_a]$$
$$= \frac{s}{l^2}[r^c(\widehat{\mathcal{L}}_{ca}\widehat{\Lambda}_b - \widehat{\mathcal{L}}_{cb}\widehat{\Lambda}_a) - r_a\widehat{\Lambda}_b + r_b\widehat{\Lambda}_a] \tag{322}$$

and so on. Expectedly, the ordinary angular momentum operator gets replaced by its covariantised version.

Matter fields may be likewise defined. The fermion current $j_{\mu\nu}$ will be defined just as the two form field,

$$\hat{j}_{ab} = K_{a\mu}K_{b\nu}j^{\mu\nu} \tag{323}$$

while the inverse relation is given by

$$j_{\mu\nu} = \left(1 + s\frac{x^2}{4l^2}\right)^{-4} K_{a\mu}K_{b\nu}\hat{j}^{ab} = \frac{\partial r_a}{\partial x^\mu}\frac{\partial r_b}{\partial x^\nu}\hat{j}^{ab} \tag{324}$$

It is simple to verify that this map preserves the form invariance of the interaction,

$$\int \mathrm{d}^4x(j_{\mu\nu}B^{\mu\nu}) = \int \mathrm{d}\Omega(\hat{j}_{ab}\widehat{B}^{ab}) \tag{325}$$

quite akin to (134).

8. Zero curvature limit

The null curvature limit (which is also equivalent to a vanishing cosmological constant) is obtained by setting $l \to \infty$. Then the A(dS) group contracts to the Poincare group so that the field theory on the A(dS) space should contract to the corresponding theory on the flat Minkwski space. This is shown very conveniently in the present formalism using Killing vectors. The example of Yang–Mills theory with sources will be considered.

The equation of motion in the A(dS) space obtained by varying the action composed of the pieces (72) and (134) is found to be,

$$s\frac{1}{2l^2}\widehat{\mathcal{L}}_{ab}\widehat{F}^{abc} + \hat{j}^c = 0 \tag{326}$$

The operator appearing in the above equation is now mapped to the flat space. The mapping for the usual angular momentum part is first derived,

$$L_{ab}\widehat{F}^{abc} = 2r_aK^\mu_b\partial_\mu\left([r^aK^{b\nu}K^{c\rho} + c.p.]F_{\nu\rho}\right) \tag{327}$$

Using the transversality condition and the identities among the Killing vectors it is seen that the only nonvanishing contribution comes from the action of the derivative on the field tensor yielding,

$$L_{ab}\widehat{F}^{abc} = 2sl^2\left(1 + s\frac{x^2}{4l^2}\right)^2 K^{c\rho}\partial^\mu F_{\mu\rho} \tag{328}$$

It is straightforward to generalise this for the covariantised angular momentum and one finds,

$$\widehat{\mathcal{L}}_{ab}\widehat{F}^{abc} = 2sl^2\left(1 + s\frac{x^2}{4l^2}\right)^2 K^{c\rho}D^{\mu}F_{\mu\rho} \tag{329}$$

Using the map (127) for the currents, the equation of motion on the A(dS) space finally gets projected on the flat space as,

$$\left(1 + s\frac{x^2}{4l^2}\right)^2 K^{c\rho}(D^{\mu}F_{\mu\rho} + \hat{j}_{\rho}) = 0 \tag{330}$$

This equation is now multiplied by the Killing vector K^{λ}_{c}. Using the identity among the Killing vectors yields,

$$\left(1 + s\frac{x^2}{4l^2}\right)^4 (D^{\mu}F_{\mu\lambda} + \hat{j}_{\lambda}) = 0 \tag{331}$$

The zero curvature limit ($l \to \infty$) is now taken. The prefactor simplifies to unity and the standard flat space Yang–Mills equation with sources is reproduced.

9. Conclusions

We have provided a manifestly covariant formulation of non-abelian interacting gauge theories defined on the A(dS) hyperboloid. The various expressions, at each step of the analysis, preserved this covariance under the appropriate kinematic groups associated with the A(dS) space. A distinctive feature was to bypass the general formulation of field theories defined on a curved space [3] in favour of exploiting the symmetry properties peculiar to the A(dS) hyperboloid. This enabled us to set up a formulation that was general enough to include both de Sitter as well as anti-de Sitter space-times, arbitrary dimensions, non-abelian gauge groups and higher rank tensor fields. Also, a complete one to one mapping with the corresponding results on a flat Minkowski space-time was established. Using this correspondence it was possible to show that in the zero cuvature limit, the A(dS) field equation passed on to the flat space field equation. This was reassuring since it is known that the groups of the A(dS) space are deformations of the Poincare group which is the kinematical group of flat Minkowiki space.

Our method consists in embedding the d-dimensional A(dS) space in a flat $(d + 1)$-dimensional space, sometimes called the ambient space. While this is a time honoured approach [34,35], we have deviated on two important issues. First, instead of working with arbitrary coordinate transformations that provide the map between the A(dS) space and the flat space, a particular type - the stereographic projection - has been used. An advantage of this is that this projection being conformal, the coordinate transformations were expressed in terms of the conformal Killing vectors. These vectors were explicitly computed by solving the Cartan-Killing equation. All expressions were thereby written in terms of the Killing vectors. Various properties of these vectors derived here were used to obtain the results compactly and transparently. The second distinctive feature was to avoid group theoretical techniques based on Casimir operators to construct the lagrangian or the action. We gave maps, involving the Killing vectors, for projecting the gauge fields on the flat space to the A(dS) space. Using these maps the action on the A(dS) space was constructed from a knowledge of the flat space action. The action so obtained was manifestly covariant under the kinematical symmetry group of the A(dS) hyperboloid. All derivatives appeared only through the angular momentum operator L_{ab}. In other approaches, apart from L_{ab}, the usual derivative ∂_a also appears. This has to be removed by choosing subsidiary conditions which are obviously not required here. Our results are in general valid for arbitrary dimensions. In particular, the expression for the axial anomaly in A(dS) space was given for any $D = 2n$ dimensions.

We analysed the Yang–Mills theory and the two form gauge theory in details. Extension to higher rank tensor fields is straightforward. Also, a discussion of the matter (fermionic) sector was provided. The Lorentz gauge fixing condition was analysed. It appeared in two equivalent versions, one of which did not have any free index (reminiscent of the usual Lorentz gauge on a flat space) while the other had a single free index. The utility of both forms was revealed. Specifically, the equivalence of our abelian gauge field equation with that obtained in other (say ambient) formalisms (after imposing appropriate subsidiary conditions) was established in the Lorentz gauge fixed sector, where both versions of the gauge fixing had to be employed.

We have computed the singlet (axial) anomaly as well as the non-abelian covariant and consistent chiral anomalies. This was done by projecting the relevant expressions from the flat to the A(dS) space. For the singlet case, we also computed the redefined expression for the axial current such that it was anomaly free. However, the current was no longer gauge invariant. This revealed the interplay between the anomaly and gauge invariance, exactly as happens for the flat example. In the non-abelian context the counterterm connecting the covariant and consistent anomalies has been calculated.

The dual field tensor was introduced from which a form of the Bianchi identity was given. Electric-magnetic duality rotations swapping this identity with the equations of motion were found.

We feel our approach gives an intuitive understanding of the closeness of formulating gauge field theories on flat and A(dS) space-time. Apart from the issues dealt here, a further application would be to develop the complete BRST formulation. This was earlier done by one of us [19] in a collaborative work for the case of the hypersphere (an n-dimansional sphere embedded in $(n+1)$-dimensional flat space). Also, a possible connection between massless and massive higher rank tensor theories with superstring theory could be envisaged.

Appendix A. Variation principle and boundary condition

The equation of motion of any system can be derived by using the variation principle according to which the action of the system is extremised. Here we will explicitly show how the equation of motion (91) comes by extremising the corresponding action (73) with suitable boundary conditions. The compatibility of these conditions on the A(dS) space and the flat space is also shown.

Varying the action (73) and using the definition for field tensor (45) we obtain,

$$\begin{aligned}\delta S &= -\frac{s}{6l^2}\int \mathrm{d}\Omega \mathrm{Tr}[\delta\widehat{F}_{abc}\widehat{F}^{abc}] = -\frac{s}{2l^2}\int \mathrm{d}\Omega \mathrm{Tr}[\delta(L_{ab}\widehat{A}_c - \mathrm{i}r_a[\widehat{A}_b,\widehat{A}_c])\widehat{F}^{abc}] \\ &= -\frac{s}{2l^2}\int \mathrm{d}\Omega \mathrm{Tr}[L_{ab}\delta\widehat{A}_c\cdot\widehat{F}^{abc} - 2ir_a[\widehat{\delta A}_b,\widehat{A}_c]\widehat{F}^{abc}]\end{aligned} \tag{332}$$

where in the last line we have used the anti-symmetric property of $\widehat{F}_{abc}$. Now,

$$\int \mathrm{d}\Omega(L_{ab}\delta\widehat{A}_c)\cdot\widehat{F}^{abc} = \int \mathrm{d}\Omega[(r_a\partial_b - r_b\partial_a)\delta\widehat{A}_c]\cdot\widehat{F}^{abc} = 2\int \mathrm{d}\Omega(\partial_b\delta\widehat{A}_c)r_a\widehat{F}^{abc} \tag{333}$$

Using the expression for invariant measure (25) we obtain,

$$\begin{aligned}\int \mathrm{d}\Omega(L_{ab}\delta\widehat{A}_c)\cdot\widehat{F}^{abc} &= 2l\int\frac{d^4r}{r_4}[(\partial_b\delta\widehat{A}_c)r_a\widehat{F}^{abc}]\\ &= 2l\int\frac{d^4r}{r_4}(\eta^{\mu}_{b}\partial_\mu\delta\widehat{A}_c)r_a\widehat{F}^{abc}\\ &= 2l\int \mathrm{d}^4r(\partial_\mu\delta\widehat{A}_c)\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\\ &= 2l\int \mathrm{d}^4r\left[\partial_\mu\left(\delta\widehat{A}_c\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\right)\right]-2l\int d^4r\left[\delta\widehat{A}_c\partial_\mu\left(\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\right)\right]\end{aligned}\tag{334}$$

Here $\partial_\mu=\frac{\partial}{\partial r^\mu}$. By Gauss' divergence theorem the first term of the above gives the surface term which vanishes at the boundary if we consider $\delta\widehat{A}_c=0$ at the boundary. Using this boundary condition the above expression simplifies to,

$$\int \mathrm{d}\Omega(L_{ab}\delta\widehat{A}_c)\ \cdot\ \widehat{F}^{abc}=-2l\int \mathrm{d}^4r\delta\widehat{A}_c\partial_\mu\left[\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\right]\tag{335}$$

Now,

$$\begin{aligned}\partial_\mu\left[\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\right] &= -\frac{1}{(r_4)^2}\frac{\partial r_4}{\partial r^\mu}r_a\widehat{F}^{a\mu c}+\frac{1}{r_4}\partial_\mu[r_\nu\widehat{F}^{\nu\mu c}+r_4\widehat{F}^{4\mu c}]\\ &= -\frac{1}{(r_4)^2}\frac{\partial r_4}{\partial r^\mu}r_a\widehat{F}^{a\mu c}+\frac{1}{r_4}[\widehat{F}^{\mu\mu c}+r_\nu\partial_\mu\widehat{F}^{\nu\mu c}+\partial_\mu r_4\ \cdot\ \widehat{F}^{4\mu c}+r_4\partial_\mu\widehat{F}^{4\mu c}]\end{aligned}\tag{336}$$

Since $\widehat{F}^{\mu\mu c}=0$ and $\frac{\partial r_4}{\partial r^\mu}=-\frac{sr_\mu}{r_4}$ the above expression reduces to,

$$\begin{aligned}\partial_\mu\left[\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\right] &= \frac{s}{(r_4)^3}r_\mu(r_\nu\widehat{F}^{\nu\mu c}+r_4\widehat{F}^{4\mu c})+\frac{1}{r_4}(r_a\partial_\mu\widehat{F}^{a\mu c}-\frac{sr_\mu}{r_4}\widehat{F}^{4\mu c})\\ &= \frac{s}{(r_4)^3}r_\mu r_\nu\widehat{F}^{\nu\mu c}+\frac{s}{(r_4)^2}r_\mu\widehat{F}^{4\mu c}+\frac{1}{r_4}r_a\partial_\mu\widehat{F}^{a\mu c}-\frac{s}{(r_4)^2}r_\mu\widehat{F}^{4\mu c}\end{aligned}\tag{337}$$

The first term vanishes as $\widehat{F}^{\nu\mu c}$ is anti-symmetric in μ and ν and second and last terms cancel each other. Hence we get,

$$\partial_\mu\left[\frac{1}{r_4}r_a\widehat{F}^{a\mu c}\right]=\frac{1}{r_4}r_a\partial_\mu\widehat{F}^{a\mu c}=\frac{1}{r_4}r_a\partial_b\widehat{F}^{abc}=\frac{1}{2r_4}L_{ab}\widehat{F}^{abc}\tag{338}$$

Substituting this in (335) we obtain,

$$\int \mathrm{d}\Omega(L_{ab}\delta\widehat{A}_c)\widehat{F}^{abc}=-\int \mathrm{d}\Omega\delta\widehat{A}_cL_{ab}\widehat{F}^{abc}\tag{339}$$

Likewise the second term of (332) can be simplified as,

$$\int d\Omega \mathrm{Tr}[-2r_a[\delta\widehat{A}_b,\widehat{A}_c]\widehat{F}^{abc}] = \int d\Omega \mathrm{Tr}[-2r_a(\delta\widehat{A}_b\widehat{A}_c - \widehat{A}_c\delta\widehat{A}_b)\widehat{F}^{abc}]$$
$$= \int d\Omega \mathrm{Tr}(-2r_a\delta\widehat{A}_b[\widehat{A}_c,\widehat{F}^{abc}])$$
$$= \int d\Omega \mathrm{Tr}(\delta\widehat{A}_c \cdot 2r_a[\widehat{A}_b,\widehat{F}^{abc}])$$
$$= \int d\Omega \mathrm{Tr}\{\delta\widehat{A}_c(r_a[\widehat{A}_b,\widehat{F}^{abc}] - r_b[\widehat{A}_a,\widehat{F}^{abc}])\} \tag{340}$$

Substituting (339) and (340) in (332) we obtain,

$$\delta S = \frac{s}{2l^2}\int d\Omega \mathrm{Tr}[\delta\widehat{A}_c\{L_{ab}\widehat{F}^{abc} - \mathrm{i}r_a[\widehat{A}_b,\widehat{F}^{abc}] + \mathrm{i}r_b[\widehat{A}_a,\widehat{F}^{abc}]\}] \tag{341}$$

Hence $\delta S = 0$ (subjected to the boundary condition $\delta\widehat{A}_a = 0$) yields,

$$\{L_{ab} - \mathrm{i}[r_a\widehat{A}_b - r_b\widehat{A}_a,]\}\widehat{F}^{abc} = 0 \tag{342}$$

i.e

$$\widehat{\mathcal{L}}_{ab}\widehat{F}^{abc} = 0 \tag{343}$$

which is the equation of motion (91).

The equivalence of this equation of motion with the stereographically projected YM equation of motion on flat space was shown in Section 8. Now the equation of motion on flat space is derived by the variation principle where we use the boundary condition, $\delta A_\mu = 0$. Again, the vector field on the A(dS) space is the projection of the flat space field through the Killing vector which we have shown in (8). Taking the variation of (8) we find, $\delta\widehat{A}_a = K^\mu_a\delta A_\mu = 0$ on the boundary which is precisely the boundary condition used in the obtention of the equation of motion on A(dS) space. Therefore we can conclude that the boundary condition on the flat space is really compatible with the boundary condition on the A(dS) space as the fields and their variations are connected by a Killing vector.

References

[1] P.A.M. Dirac, Ann. Math 36 (1935) 657.
[2] S. Perlmutter et al., Astrophys. J. 517 (1999) 565, astro-ph/9812133;
S. Hannay et al., Astrophys. J. Lett. 545 (2000) 5.
[3] R.M. Wald, Quantum Fields Theory in Curved Spacetime and Black Hole Thermodynamics, The University of Chicago Press, 1994.
[4] T. Biswas, W. Siegel, JHEP 0207 (005) (2002), hep-th/0203115.
[5] K. Hallowell, A. Waldron, Nucl. Phys. B 724 (2005) 453 (PM).
[6] R.R. Metsaev, Arbitrary spin massless bosonic fields in *d*-dimensional anti-de Sitter space, hep-th/9810231.
[7] A. Fotopoulos, K.L. Panigrahi, M. Tsulaia, Phys. Rev. D 74 (2006) 085029, hep-th/0607248.
[8] J. Bros, J.P. Gazeau, U. Moschella, Phys. Rev. Lett. 73 (1994) 1746.
[9] B. Bros, U. Moschella, Rev. Math. Phys. 8 (1996) 327.
[10] S. Rouhani, M.V. Takook, Abelian Gauge Theory in de Sitter Space, gr-qc/0502019.
[11] T. Garidi, J.-P.Gazeau, S. Rouhani, M.V. Takook, Massless Vector Field in de Sitter Universe, gr-qc/0608004.
[12] C.G. Callan, F. Wilczek, Nucl. Phys. B 340 (1990) 366.
[13] A. Strominger, JHEP 0110 (2001) 034, hep-th/0106113.
[14] L. Dyson, J. Lindesay, L. Susskind, JHEP 0208 (2002) 045. hep-th/0202163.

[15] J. Maldacena, Adv. Theor. Math. Phys. 2 (231) (1998), hep-th/9711200.
[16] J.L. Petersen, Introduction to the Maldacena Conjecture on AdS/CFT, hep-th/9902131.
[17] R. Banerjee, Gauge Theories on de Sitter Space and Killing Vectors, Annals of Phys. 322 (2007) 2129. hep-th/0608045.
[18] R. Banerjee, Ann. Phys. 311 (2004) 245, hep-th/0307296.
[19] R. Banerjee, S. Deguchi, Phys. Lett. B 632 (2006) 579, hep-th/0509161.
[20] F. Gursey (Ed.), Group Theoretical Concepts and Methods in Elementary Particle Physics, Istanbul Summer School of Theoretical Physics, Gordon and Breach, New York, 1962.
[21] See for example chapter 4 P. Di Francesco, P. Mathieu, D. Senechal, Conformal Field Theory, Springer-Verlag, New York, 1997.
[22] S·L. Adler, Phys. Rev. D 6 (1972) 3445.
[23] M. Blau, JHEP 0011 (2000) 023, hep-th/0005098.
[24] R. Jackiw, C. Rebbi, Phys. Rev. D 14 (1976) 517.
[25] S. Adler, Phys. Rev. 177 (1969) 2426;
J. Bell, R. Jackiw, Nuovo Cimento 60A (1969) 47.
[26] K. Fujikawa, Phys. Rev.Lett. 42 (1979) 1195;
K. Fujikawa, Phys. Rev. D 21 (1980) 2848.
[27] P.J. O'Donnell, B. Wong, Phys. Lett. B 138 (1984) 274.
[28] W.A. Bardeen, B. Zumino, Nucl. Phys. B 244 (1984) 421–453.
[29] H. Banerjee, R. Banerjee, Phys. Lett. B 174 (1986) 313;
H. Banerjee, R. Banerjee, P. Mitra, Z. Phys. C 32 (445) (1986).
[30] K. Fujikawa, H. Suzuki, Path Integrals and Quantum Anomalies, Clarendon Press, Oxford, 2004.
[31] R.A. Bertlmann, Anomalies in Quantum Field Theory, OUP, Oxford, 2000.
[32] M. Kalb, P. Ramond, Phys. Rev. D 9 (1974) 2273;
E. Cremmer, J. Scherk, Nucl. Phys. B 72 (1974) 117.
[33] D.G.C. McKeon, Antisymmetric tensor gauge fields on S^4, (arXiv:hep-th/0305009).
[34] M. Gutzwiller, Helv. Phys. Acta 29 (1956) 313.
[35] S. Deser, R.I. Nepomechie, Ann. Phys. 154 (1984) 396.