Analyzing jump phenomena and stability in nonlinear oscillators using renormalization group arguments

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We study the stability of a damped Duffing oscillator by employing a renormalization group method for solving nonlinear differential equations. This approach is direct and makes the study of the amplitude equation smooth, obvious, and not based on any initial *ansatz* for the periodic form of the final solution. An introduction to the renormalization method is given.

I. INTRODUCTION

The inclusion of nonlinear effects in periodically driven nonlinear oscillators[1] yields interesting behavior.[2–4] For example, a plot of the amplitude versus the difference of the frequencies between the external and the natural frequency of the oscillator shows sudden jumps at certain threshold values of the frequency difference, a phenomenon similar to hysteresis in ferromagnetism.

Studying differential equations from the point of view of renormalization group symmetries is an interesting field of research.[5–7] In this method an initial *ansatz*[8–10] for the form of the final solution is not required. In this paper we study the jumps of the amplitude of the forced nonlinear (Duffing) oscillator using the renormalization group approach. Because jump phenomena are well known in the nonlinear dynamics literature, we start with the end result.

The system of interest is a nonlinear oscillator subjected to damping and periodic forcing. In Fig. 1 we plot the amplitude of the oscillator as a function of the difference between the external frequency and the natural frequency of the oscillator. As the frequency is increased, the amplitude increases along BC and then makes a jump from C to E. If the frequency is then decreased, the amplitude takes a different route and goes through ED and then jumps from D to B, thus always avoiding CD. This behavior is the well known jump phenomenon.

Before studying this phenomenon in detail, we introduce the simpler case of a quartic oscillator (a Duffing oscillator without forcing and damping). This example will help us obtain a feel of how the renormalization group method works in the present context.

II. THE QUARTIC OSCILLATOR: A COMPARATIVE STUDY OF METHODS

The potential governing a quartic oscillator is

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{\lambda m}{4}x^4, \tag{1}$$

which has the equation of motion

$$\ddot{x} = -\omega^2 x - \lambda x^3. \tag{2}$$

The motion is bounded and periodic with the energy of the oscillator given by

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 + \frac{\lambda m}{4}x^4 \tag{3a}$$

$$= \frac{1}{2}m\omega^2 a^2 + \frac{\lambda m}{4}a^4, \tag{3b}$$

where a denotes the amplitude of the oscillation. We equate these two equivalent forms of E and obtain

$$\dot{x} = \frac{dx}{dt} = \omega\sqrt{(a^2 - x^2) + \frac{\lambda}{2\omega^2}(a^4 - x^4)}. \tag{4}$$

Equation (4) directly leads to an integral expression for the period

$$T = 4\frac{1}{\omega}\int_0^a \frac{dx}{\sqrt{(a^2 - x^2) + \frac{\lambda}{2\omega^2}(a^4 - x^4)}} \tag{5a}$$

$$= \frac{4}{\omega}\int_0^1 \frac{dy}{\sqrt{1-y^2}\left[1 + \frac{\lambda a^2}{2\omega^2}\frac{1-y^4}{1-y^2}\right]^{1/2}}, \tag{5b}$$

where we have made the substitution $y = x/a$. The integral in Eq. (5b) cannot be evaluated exactly. For small λ the period is given by

$$T = \frac{4}{\omega}\int_0^1 \frac{dy}{\sqrt{1-y^2}}\left[1 - \frac{\lambda a^2}{4\omega^2}\frac{1-y^4}{1-y^2}\right], \tag{6}$$

where we have retained terms up to $O(\lambda)$. If we substitute $y = \sin\alpha$, we can do the integral and find

$$T = \frac{2\pi}{\omega}\left(1 - \frac{3a^2\lambda}{8\omega^2}\right). \tag{7}$$

Thus, due to the nonlinearity of the potential, the period is changed from its harmonic potential value $2\pi/\omega$. We will now obtain the same result using a multiple time scale analysis and the renormalization group method and then explore the link between them.

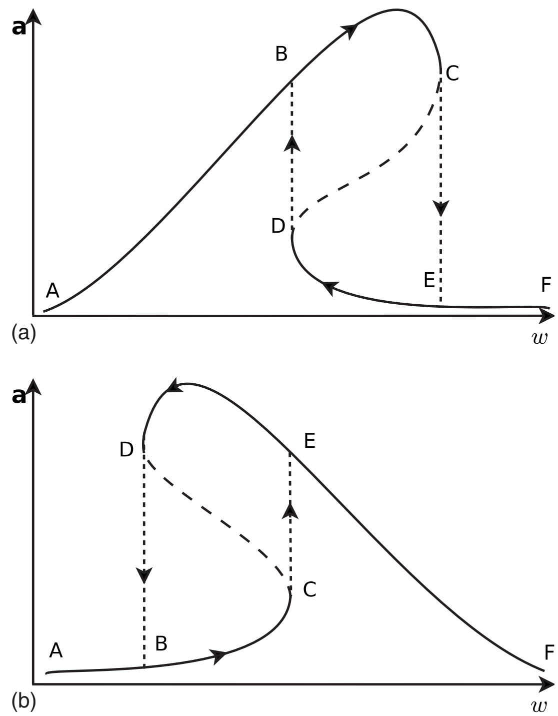

Fig. 1. Plot of frequency versus amplitude. ω is the difference between the forcing and natural frequencies, and a is the amplitude. (a) The soft spring, corresponding to the "M" shaped quartic potential, that is, $V(x)=Cx^2-Dx^4$. (b) The hard spring corresponding to the quartic trap potential, that is, $V(x)=Cx^2+Dx^4$. C and D are positive constants.

Perturbation theory in its most popular form starts with the expansion

$$x(t)=x_0(t)+\lambda x_1(t)+\ \dots\ , \tag{8}$$

where λ is now being used as the perturbation parameter. We substitute this expansion in Eq. (2) and obtain

$$\frac{\partial^2 x_0}{\partial t^2}+\omega^2 x_0=0 \quad (\lambda^0), \tag{9}$$

$$\frac{\partial^2 x_1}{\partial t^2}+\omega^2 x_1=-x_0^3 \quad (\lambda^1). \tag{10}$$

When the solution of Eq. (9), $x=a\cos(\omega t+\theta)$, is used on the right-hand side of Eq. (10), we obtain

$$\ddot{x}_1+\omega^2 x_1=-a^3\cos^3(\omega t+\theta) \tag{11a}$$

$$=-\frac{a}{4}\cos 3(\omega t+\theta)-\frac{3a^3}{4}\cos(\omega t+\theta). \tag{11b}$$

If the right-hand side of Eq. (11) were zero, the solution would be a cosine function as it is for Eq. (9). As we shall see, the presence of the resonating cosine term on the right-hand side of Eq. (11) is the reason for the appearance of the secular (divergent) term in the solution. The goal is to eliminate these divergences.

We write the solution of Eq. (9) as $x_0=A\cos t+B\sin t=a\cos(\omega t+\theta)$. The Wronskian is $W=AB\omega$. The particular solution of Eq. (11) becomes

$$x_1=A\cos\omega t\int dt\frac{-B\sin\omega t}{AB\omega}[-a^3\cos^3(\omega t+\theta)]$$
$$+B\sin\omega t\int dt\frac{A\cos\omega t}{AB\omega}[-a^3\cos^3 a(\omega t+\theta)] \tag{12}$$

$$=-\frac{3a^3}{16\omega^2}\cos(\omega t+\theta)-\frac{a^3}{32\omega^2}\cos 3(\omega t+\theta)$$
$$-t\frac{3a^3}{8\omega}\sin(\omega t+\theta). \tag{13}$$

The last term in Eq. (13) diverges linearly in t. The full solution (up to first order in λ) from Eq. (8) is

$$x=a\cos(\omega t+\theta)+\lambda\left[-\frac{3a^3}{16\omega^2}\cos(\omega t+\theta)\right.$$
$$\left.-\frac{a^3}{32\omega^2}\cos 3(\omega t+\theta)\right]-\frac{3a^3}{8\omega}\lambda t\sin(\omega t+\theta). \tag{14}$$

For pedagogical reasons we introduce the method of multiple time scales based on the following observations. In Eq. (7) the correction to the unperturbed time period $(2\pi/\omega)$ is of the order of λ, that is, this correction belongs to a much slower time scale. Equation (14) implies that for times $t\ll 1/\lambda$, the divergence is not prominent. But as t becomes $\sim 1/\lambda$ or longer, the secular divergence becomes significant, leading to an unphysical solution.

The method of multiple time scales uses two separate independent time scales equal to $t_0=\lambda^0 t=t$ and $t_1=\lambda^1 t=\lambda t=\lambda t_0$. What takes large values on the faster time scale t_0 takes much lower values (to first order) in the slower time scale t_1. For example, what requires, say, 300 time units (seconds) in the fast time scale requires only five time units (minutes) on the slower time scale. Because both time scales obey the same evolution Eq. (2), the idea of the method of multiple time scales is to express x as a function of both t_0 and t_1 (the method can be generalized to any $t_n=\lambda^n t$). We write

$$x=x(t_0,t_1), \tag{15}$$

$$\frac{dx}{dt}=\frac{\partial x}{\partial t_0}+\frac{\partial x}{\partial t_1}\frac{\partial t_1}{\partial t}=\frac{\partial x}{\partial t_0}+\lambda\frac{\partial x}{\partial t_1}, \tag{16}$$

$$\frac{d^2x}{dt^2}=\frac{\partial^2 x}{\partial t_0^2}+2\lambda\frac{\partial^2 x}{\partial t_1\,\partial t_0}+O(\lambda^2). \tag{17}$$

We also write x in Eq. (8) as

$$x(t_0,t_1)=x_0(t_0,t_1)+\lambda x_1(t_0,t_1)+\dots \tag{18}$$

and use this expansion in Eq. (2) to obtain to zeroth order

$$\frac{\partial^2 x_0}{\partial t_0^2}+\omega^2 x_0=0, \tag{19}$$

which yields

$$x_0=a\cos(\omega t_0+\theta). \tag{20}$$

On the right-hand side of Eq. (20), a and θ can both be functions of (t_0,t_1). To first order in λ we have

$$\frac{\partial^2 x_1}{\partial t_0^2}+\omega^2 x_1+2\frac{\partial^2 x}{\partial t_1\,\partial t_0}=-x_0^3=-a^3\cos^3(\omega t_0+\theta). \tag{21}$$

We use $\partial x_0/\partial t_0 = -a\omega \sin(\omega t_0+\theta)$ and

$$\frac{\partial}{\partial t_1}\frac{\partial x_0}{\partial t_0} = -\frac{\partial a}{\partial t_1}\omega \sin(\omega t_0+\theta) - a\omega \cos(\omega t_0+\theta)\frac{\partial \theta}{\partial t_1} \quad (22)$$

in Eq. (21) and obtain

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 - 2\left[\frac{\partial a}{\partial t_1}\omega \sin(\omega t_0+\theta) + a\omega \cos(\omega t_0 + \theta)\frac{\partial \theta}{\partial t_1}\right] = -\frac{a^3}{4}[3\cos(\omega t_0+\theta) + \cos 3(\omega t_0+\theta)]. \quad (23)$$

Equating the coefficients of the various harmonics yields

$$\frac{\partial a}{\partial t_1} = 0, \quad \frac{\partial \theta}{\partial t_1} = \frac{3a^2}{8\omega} \quad (24)$$

and

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = -\frac{a^3}{4}\cos 3(\omega t_0+\theta). \quad (25)$$

These conditions lead to a=constant and $\theta=(3a^2/8\omega)t_1$; x_1 can be evaluated directly from Eq. (25). The final expression for x, according to Eq. (18), takes the form,

$$x = a\cos\left(\omega t_0 + \frac{3a^2}{8\omega}t_1\right) + \lambda x_1(t_0,t_1) + \ldots \quad (26a)$$

$$= a\cos\left(\omega + \frac{3a^2}{8\omega}\lambda\right)t_0 + \lambda x_1(t_0,t_1) + \ldots \quad (26b)$$

$$= a\cos \omega\left(1 + \frac{3a^2\lambda}{8\omega^2}\right)t + \lambda x_1(t_0,t_1) + \ldots . \quad (26c)$$

The most striking feature of Eq. (26) is that the corrected frequency $\omega(1+3a^2\lambda/8\omega^2)$ yields the correct period (to first order in λ), which is equal to $(2\pi/\omega)(1+3\lambda a/8\omega^2)^{-1} = (2\pi/\omega)(1-3\lambda a^2/8\omega^2)$, in agreement with Eq. (7).

We now apply the renormalization group method to the oscillator. In the derivation from Eq. (8) to Eq. (14), the choice of $t=0$ is arbitrary. Although such a statement is obvious, it is worth going a bit deeper into its meaning. Suppose person 1 studies the dynamics on the seconds' scale, which for now, we call the "standard" scale. We next fix our attention on the instant of time corresponding to the event occurring at $t=300$ on this scale. Person 2 starts his/her study of the same dynamics by choosing a different scale and setting his/her $t=0$ at the instant of time corresponding to the 100 on the standard scale. Then, after a time interval (which is equal to $300-100=200$ in the standard scale), person 2 observes the same event. Similarly, person 3 chooses yet another scale and starts at, say, 131.25 on the standard scale. This person would find the same observation after a duration equal to $300-131.25=168.75$ in the standard scale from her/his starting point. Whatever the starting time and the units in the different scales chosen by different observers, all will observe the same event at the particular time instant t. In this way, we can envisage an infinite set of time scales, all leading to the same observation at the time t.

Generating an infinite set of time scales by using the flexibility in the choice of $t=0$ is the key feature of the renormalization group method. We introduce an arbitrary time scale μ and write for t in the divergent part of Eq. (13)

$$t = [t-\mu] + [\mu - 0]. \quad (27)$$

As we shall see, the first pair of brackets encloses the "present" and the latter pair of brackets encloses the "past." Our aim will be to place this μ at such a point on the interval $(0\to t)$ so that all divergences go to the past, and the present time scale remains divergence free. The essential steps of the method can be summarized as follows.

(1) The amplitude and phase, which are the quantities to be renormalized, are written as

$$a(0) = a_0 = Z_1(\mu)a(\mu), \quad (28)$$

$$\theta(0) = \theta_0 = \theta(\mu) + Z_2(\mu), \quad (29)$$

where

$$Z_1(\mu) = 1 + A_1\lambda + \ldots , \quad (30)$$

$$Z_2(\mu) = B_1\lambda + \ldots \quad (31)$$

are renormalization constants to be calculated perturbatively.

(2) Equation (14) becomes

$$x = a(\mu)(1+A_1\lambda)\cos(\omega t + \theta(\mu) + B_1\lambda) + \lambda\left[-\frac{3}{16\omega^2}a^3(\mu)\cos(\omega t+\theta(\mu)) - \frac{1}{32\omega^2}a^3(\mu)\cos 3(\omega t + \theta(\mu)) - \frac{3t}{8\omega}a^3(\mu)\sin(\omega t+\theta(\mu))\right] \quad (32a)$$

$$= a(\mu)\cos(\omega t+\theta(\mu)) + \lambda\left[\left\{ a(\mu)A_1 - \frac{3a^3(\mu)}{16\omega^2}\right\}\cos(\omega t+\theta(\mu)) - a(\mu)B_1 - t\frac{3a^3(\mu)}{8\omega}\right\}\sin(\omega t + \theta(\mu)) - \frac{a^3(\mu)}{32\omega^2}\cos 3(\omega t+\theta(\mu))\right]. \quad (32b)$$

(3) Equation (27) is invoked to account for the divergence in Eq. (32b), and A_1 and B_1 are chosen to connect μ and 0. The divergent parts are put into the $[\mu-0]$ part of Eq. (27), and the well behaved part is retained in the $[t-\mu]$ part. From the coefficients of $\cos(\omega t+\theta(\mu))$ and $\sin(\omega t + \theta(\mu))$ in the λ-term of Eq. (32b), we obtain

$$A_1 = 0, \quad (33)$$

$$B_1 = \mu\frac{3a^2(\mu)}{8\omega}. \quad (34)$$

(4) Now that the renormalized constants are determined (to first order in λ) and the divergence eliminated through Eq. (34), we are left with

$$x = a(\mu)\cos(\omega t + \theta(\mu)) + \lambda\Bigg[-\frac{3a^3(\mu)}{16\omega^2}\cos(\omega t + \theta(\mu)) - \frac{a^3(\mu)}{32\omega^2}\cos 3(\omega t + \theta(\mu)) - (t-\mu)\frac{3a^3(\mu)}{8\omega}\sin(\omega t + \theta(\mu))\Bigg]. \quad (35)$$

Whatever our choice of μ, the dynamics is independent of μ. Thus, we set $dx/d\mu=0$ and separately collect the coefficients of the cosine and sine terms and obtain

$$\frac{da}{d\mu} = 0, \quad (36)$$

$$\frac{d\theta}{d\mu} = \frac{3a^2\lambda}{8\omega}, \quad (37)$$

which imply that a is a constant and $\theta=\mu 3a^2\lambda/8\omega$. If we use Eqs. (36) and (37), x becomes

$$x = a\cos\left(\omega t + \mu\frac{3a^2\lambda}{8\omega}\right) + \lambda\Bigg[-\frac{3a^3}{16\omega^2}\cos(\omega t + \theta(\mu)) - \frac{a^3}{32\omega^2}\cos 3(\omega t + \theta(\mu)) - (t-\mu)\frac{3a^3}{8\omega}\sin(\omega t + \theta(\mu))\Bigg]. \quad (38)$$

(5) Because μ can take any value from 0 to t, we now merge μ with t to remove the residual divergence in the $[t-\mu]$ term of Eq. (38). Hence the final step involves setting $\mu=t$ in Eq. (38), yielding

$$x = a\cos\left(\omega + \frac{3a^2\lambda}{8\omega}\right)t - \lambda\left[\frac{3a^3}{16\omega^2}\cos(\omega t + \theta) + \frac{a^3}{32\omega^2}\cos 3(\omega t + \theta)\right]. \quad (39)$$

We have thus renormalized the solution to first order in λ. It is satisfying to obtain in Eq. (39) the same correction to the frequency as in Eq. (26c) or Eq. (7). Note that multiple time scales and renormalization group methods involve different ways of viewing the time interval 0 and t. The former attempts to contain the secular divergence by going to a larger time scale. We keep the limits of the interval 0 and t fixed and introduce several time scales within this interval, with the unit step size of the scales being multiples (in powers of the reciprocal of the small perturbation parameter λ) of the fastest scale in the system. The renormalization group method lifts the restriction of keeping the duration of the time interval fixed. It works in one single but arbitrary time scale. This arbitrariness is all important. We require that the result at time t is independent of this arbitrary time scale. Whatever has happened in the past, i.e., when it has started measuring or how long its time step-length has been, are irrelevant. This way of thinking amounts to splitting the interval of $0-t$ arbitrarily into two parts, putting all the divergences in the past, eliminating the divergences by defining constants accordingly, and then choosing this arbitrarily introduced time scale in such a way that its zero always is the running time t itself (i.e., $\mu=t$).

This method is analogous to the way the renormalization group method is used to eliminate the faster Fourier modes in the critical phenomena. In critical phenomena we split the momentum range (zero to some upper cut-off value Λ) into two parts $(0,\Lambda/b,\Lambda)(b>1)$ and then integrate out the Fourier components between Λ/b and Λ (in analogy with pushing all the divergences to the past), thus reducing the number of degrees of freedom. Finally we scale the momenta by the same factor b so that the cut-off is again set to its original value Λ (in analogy with merging μ with t). For a recent work on the renormalization group method, we refer the reader to Ref. 11.

Perturbation theory, when used to treat nonlinear terms, frequently breaks down by exhibiting unwanted divergences. The cure lies in formulating the right perturbation expansion; this process is called renormalization. A beautiful introduction to the subject is given in Ref. 12. Usually, the coupling constant for the perturbation (and also other expansion parameters describing the theory) is the culprit, and we need to redefine the coupling constants to make sense of the perturbation theory. This process often leads to a scale dependent coupling constant with an arbitrary scale. Because the physical quantities cannot depend on this arbitrary scale, we arrive at the renormalization group equations.[6] What was shown in Ref. 6 a little more than a decade ago was that a similar renormalization group procedure can be needed on a class of differential equations. The parameters of the theory, which are inaccessible at a given time, are the initial value or the Cauchy data. They showed that the divergent perturbation theory can be made finite by renormalizing the Cauchy data, which are now dependent on an arbitrary time scale. Removing the dependence on the arbitrary time scale from a physical quantity leads to the renormalization group equations.

III. THE JUMP PHENOMENON

We now apply the perturbative renormalization group method to the jump phenomenon. The equation of motion of the oscillator, which takes into account forcing and damping, is

$$\ddot{x} + \omega_0^2 x + k\dot{x} - \beta x^3 = F\cos\Omega t. \quad (40)$$

We will see that forcing, nonlinearity, and damping all have significant roles t in the jump phenomenon. If we set $\Omega t=\tau$ and divide by Ω^2, we can express Eq. (40) as

$$\ddot{x} + \omega_r^2 x + k_r\dot{x} - \lambda x^3 = \bar{F}\cos\tau, \quad (41)$$

where the rescaled terms are $\omega_r=\omega_0/\Omega$, $k_r=k/\Omega^2$, $\lambda=\beta/\Omega^2$, and $\bar{F}=F/\Omega^2$. Our goal is to study the near resonance scenario. To this end we follow the procedure in Ref. 8, except that we introduce four perturbation parameters. They are ϕ, α, ϵ, and λ, in terms of which we write $\omega_r^2=1+\phi\omega$, $\bar{F}=\alpha f$, and $k_r=\epsilon\chi$.

These parameters are independent of each other: ϵ is relevant to the damping, λ is the nonlinearity, ϕ is the shift of the external frequency from the natural one, and α is relevant to the magnitude of the forcing. We introduced several parameters to keep track of the separate roles of damping, nonlinearity, and forcing. When we obtain the final result, we will see that all scaling relations (which are arbitrary) are automatically adjusted to yield the correct amplitude equation. For example, if there were a mixture of multiple powers instead of a single x^3 term on the left-hand side of Eq. (41), then the renormalization group method would still work and give us the same amplitude equation as might be obtained by other methods. The reason lies in the arbitrariness of μ. There is no multiplicative operation in dividing the time interval, that is, μ is not a factor of t.

We rewrite Eq. (41) as

$$\ddot{x} + (1+\phi\omega)x + \epsilon\chi\dot{x} - \lambda x^3 = \alpha f \cos\tau \tag{42}$$

and expand x in terms of the four parameters,

$$x = x_0 + \epsilon x_{1\epsilon} + \lambda x_{1\lambda} + \phi x_{1\phi} + \alpha x_{1\alpha} + \text{higher order terms}. \tag{43}$$

Equations (42) and (43) imply a perturbation theory without a restrictive initial ansatz. We equate the first powers of the expansion parameters separately and obtain the following equations:

$$\ddot{x}_0 + x_0 = 0 \quad (\text{zeroth order}), \tag{44}$$

$$\ddot{x}_{1\epsilon} + x_{1\epsilon} = -\chi\dot{x}_0 \quad (\epsilon^1), \tag{45}$$

$$\ddot{x}_{1\lambda} + x_{1\lambda} = x_0^3(\lambda^1), \tag{46}$$

$$\ddot{x}_{1\phi} + x_{1\phi} = -\omega x_0(\phi^1), \tag{47}$$

$$\ddot{x}_{1\alpha} + x_{1\alpha} = f\cos\tau(\alpha^1). \tag{48}$$

The solution of Eq. (44) is

$$x_0 = a\cos(\tau+\theta). \tag{49}$$

In this approach we do not assume any time dependence for a and θ, and thus we can integrate the other differential equations. Rewriting Eq. (45) as $\ddot{x}_{1\epsilon}+x_{1\epsilon}=\chi\sin(\tau+\theta)$ yields

$$x_{1\epsilon} = \frac{1}{2}\chi a\left[\frac{1}{2}\sin(\tau+\theta) - \tau\cos(\tau+\theta)\right]. \tag{50}$$

The last term has a secular divergence and with increasing time would give an unbounded solution. Similarly, Eqs. (46)–(48) yield

$$x_{1\lambda} = -\frac{a^3}{32}\cos 3(\tau+\theta) + \frac{3a^3}{16}\cos(\tau+\theta) + \tau\frac{3a^3}{8}\sin(\tau+\theta), \tag{51}$$

$$x_{1\phi} = -\frac{\omega a}{4}\cos(\tau+\theta) - \tau\frac{\omega a}{2}\sin(\tau+\theta), \tag{52}$$

$$x_{1\alpha} = \frac{f}{4}\cos\tau + \tau\frac{f}{2}\sin\tau. \tag{53}$$

In Eqs. (50)–(53) the last terms carry the secular divergences. These divergences appear independent of the initial time (here $\tau=0$). Our goal is to capitalize on this freedom in choice of the initial time. The final expression for x (up to first order) is

$$x = x_0 + \epsilon x_{1\epsilon} + \lambda x_{1\lambda} + \phi x_{1\phi} + \alpha x_{1\alpha} \tag{54}$$

$$\begin{aligned} &= a\cos(\tau+\theta) + \epsilon\left[\frac{1}{2}\chi a\left\{-\tau\cos(\tau+\theta) + \frac{1}{2}\sin(\tau+\theta)\right\}\right] \\ &\quad + \lambda\left[-\frac{a^3}{32}\cos 3(\tau+\theta) + \frac{3a^3}{16}\cos(\tau+\theta) + \tau\frac{3a^3}{8}\sin(\tau+\theta)\right] \\ &\quad + \phi\left[-\frac{\omega a}{4}\cos(\tau+\theta) - \tau\frac{\omega a}{2}\sin(\tau+\theta)\right] \\ &\quad + \alpha\left[\frac{f}{4}\cos\tau + \tau\frac{f}{2}\sin\tau\right]. \end{aligned} \tag{55}$$

To treat the divergent (secular) terms we reparametrize a and θ by introducing renormalizing constants Z_1 and Z_2. They will be calculated perturbatively as follows:

$$a = a(\mu)Z_1(\mu) = a(\mu)(1 + \epsilon Z_{1\epsilon} + \lambda Z_{1\lambda} + \phi Z_{1\phi} + \alpha Z_{1\alpha} + \dots), \tag{56}$$

$$\theta = \theta(\mu) + Z_2(\mu) = \theta(\mu) + \epsilon Z_{2\epsilon} + \lambda Z_{2\lambda} + \phi Z_{2\phi} + \alpha Z_{2\alpha} + \dots. \tag{57}$$

Because the solution given by Eq. (55) depends on the choice of the initial conditions, we make a and θ (which depend on the initial conditions) functions of some arbitrary time scale μ.[6] In the expression for x in Eq. (55), all the divergent terms are either of the form $\tau\cos(\tau+\theta)$ or $\tau\sin(\tau+\theta)$, except one that occurs in the α portion and has the form $\tau\sin\tau$, which we express as

$$\begin{aligned} \tau\sin\tau &= \tau\sin(\tau+\theta-\theta) \\ &= \sin(\tau+\theta)\cos\theta - \cos(\tau+\theta)\sin\theta \end{aligned} \tag{58a}$$

$$\begin{aligned} &= \sin(\tau+\theta(\mu))\cos\theta(\mu) - \cos(\tau+\theta(\mu))\sin\theta(\mu) \\ &\quad + \text{higher order terms}, \end{aligned} \tag{58b}$$

where we have invoked Eqs. (56) and (57). We use Eqs. (56) and (57) for a and θ and obtain an expression for x up to first order in the perturbation parameters as

$$\begin{aligned} x &= a(\mu)\cos(\tau+\theta(\mu)) + \epsilon\left[\left\{-\frac{\tau}{2}\chi a(\mu) + a(\mu)Z_{1\epsilon}\right\}\cos(\tau+\theta(\mu)) + \left\{\frac{\chi}{4}a(\mu) - a(\mu)Z_{2\epsilon}\right\}\sin(\tau+\theta(\mu))\right] \\ &\quad + \lambda\left[\left\{\frac{3a^3(\mu)}{16} + a(\mu)Z_{1\lambda}\right\}\cos(\tau+\theta(\mu)) + \left\{\tau\frac{3a^3(\mu)}{8} - a(\mu)Z_{2\lambda}\right\}\sin(\tau+\theta(\mu)) - \frac{a^3(\mu)}{32}\cos 3(\tau+\theta(\mu))\right] \\ &\quad + \phi\left[\left\{-\frac{\omega a(\mu)}{4} + a(\mu)Z_{1\phi}\right\}\cos(\tau+\theta(\mu)) - \left\{\tau\frac{\omega a(\mu)}{2} + a(\mu)Z_{2\phi}\right\}\sin(\tau+\theta(\mu))\right] + \alpha\left[\left\{a(\mu)Z_{1\alpha} - \tau\frac{f}{2}\sin\theta(\mu)\right\}\right. \\ &\quad \left.\times\cos(\tau+\theta(\mu)) + \left\{\tau\frac{f}{2}\cos\theta(\mu) - a(\mu)Z_{2\alpha}\right\}\sin(\tau+\theta(\mu)) + \frac{f}{4}\cos\tau\right]. \end{aligned} \tag{59}$$

We have rearranged the first order terms coming from the $x_0 = a\cos(\tau+\theta)$ term of Eq. (55) into the respective brackets and have grouped the cosine and sine terms accordingly. Explicitly,

$$
\begin{aligned}
a\cos(\tau+\theta) &= a(\mu)[1+\epsilon Z_{1\epsilon}+\lambda Z_{1\lambda}+\phi Z_{1\phi} \\
&\quad +\alpha Z_{1\alpha}]\cos[\tau+\theta(\mu)+\epsilon Z_{2\epsilon}+\lambda Z_{2\lambda}+\phi Z_{2\phi} \\
&\quad +\alpha Z_{2\alpha}] = a(\mu)(1+\epsilon Z_{1\epsilon}+\lambda Z_{1\lambda}+\phi Z_{1\phi} \\
&\quad +\alpha Z_{1\alpha})\cos(\tau+\theta(\mu)) - a(\mu)(\epsilon Z_{2\epsilon}+\lambda Z_{2\lambda} \\
&\quad +\phi Z_{2\phi}+\alpha Z_{2\alpha})\sin(\tau+\theta(\mu)) \\
&\quad + \text{higher order terms}.
\end{aligned} \tag{60}
$$

Because μ is an arbitrary time scale, we replace τ in all the divergent terms by

$$\tau = [\tau-\mu]+\mu. \tag{61}$$

The divergences now occur in terms containing μ only, which we shall use to define the Z functions.

Our next goal is to remove the divergences. From Eqs. (59) and (61) we obtain the following set of equations, which fix the renormalization constants. In the ϵ expression, the divergence in the coefficient of $\cos(\tau+\theta(\mu))$ is removed if $Z_{1\epsilon}=\frac{1}{2}\mu\chi$. For $\sin(\tau+\theta(\mu))$ there is no divergent coefficient, and hence $Z_{2\epsilon}=0$. Similarly, from the λ expression, we obtain $Z_{1\lambda}=0$ and $Z_{2\lambda}=\mu 3a^2(\mu)/8$. The ϕ and α expressions yield $Z_{1\phi}=0$, $Z_{2\phi}=\omega\mu/2$, $Z_{1\alpha}=(\mu f/2)(\sin\theta(\mu))/a(\mu)$, and $Z_{2\alpha}=(\mu f/2)(\cos\theta(\mu))/a(\mu)$, respectively. We absorb the $(\tau-\mu)$ terms in the renormalized $a(\mu)$ and $\theta(\mu)$ and find

$$
\begin{aligned}
x &= a(\mu)\cos(\tau+\theta(\mu)) + \epsilon\left[-\frac{\chi a(\mu)}{2}(\tau-\mu)\cos(\tau+\theta(\mu)) + \frac{\chi a(\mu)}{4}\sin(\tau+\theta(\mu))\right] \\
&\quad + \lambda\left[(\tau-\mu)\frac{3a^3(\mu)}{8}\sin(\tau+\theta(\mu)) + \frac{3a^3(\mu)}{16}\cos(\tau+\theta(\mu)) - \frac{a^3(\mu)}{32}\cos 3(\tau+\theta(\mu))\right] \\
&\quad + \phi\left[-\frac{1}{2}(\tau-\mu)\omega a(\mu)\sin(\tau+\theta(\mu)) - \frac{1}{4}\omega a(\mu)\cos(\tau+\theta(\mu))\right] \\
&\quad + \alpha\left[(\tau-\mu)\frac{f}{2}\{\cos\theta(\mu)\sin(\tau+\theta(\mu)) - \sin\theta(\mu)\cos(\tau+\theta(\mu))\} + \frac{f}{4}\cos\tau\right].
\end{aligned} \tag{62}
$$

Because μ is arbitrary, we require that x be independent of μ. By retaining derivatives only up to first order we obtain, $dx/d\mu=0$ and

$$
\begin{aligned}
&\frac{da}{d\mu}\cos(\tau+\theta(\mu)) - a(\mu)\frac{d\theta}{d\mu}\sin(\tau+\theta(\mu)) \\
&\quad + \epsilon\left[\frac{\chi a(\mu)}{2}\cos(\tau+\theta(\mu))\right] \\
&\quad + \lambda\left[-\frac{3a^3(\mu)}{8}\sin(\tau+\theta(\mu))\right] \\
&\quad + \phi\left[\frac{1}{2}\omega a(\mu)\sin(\tau+\theta(\mu))\right] \\
&\quad + \alpha\left[\frac{f}{2}\{\sin\theta(\mu)\cos(\tau+\theta(\mu)) - \cos\theta(\mu)\sin(\tau \right. \\
&\quad \left. + \theta(\mu))\}\right] = 0.
\end{aligned} \tag{63}
$$

If we equate the coefficients of $\cos(\tau+\theta(\mu))$ and $\sin(\tau+\theta(\mu))$ separately to zero, we find

$$\frac{da}{d\mu} = -\frac{1}{2}\epsilon\chi a - \frac{1}{2}\alpha f\sin\theta(\mu) = F_1(a,\theta), \tag{64}$$

$$\frac{d\theta}{d\mu} = -\lambda\frac{3a^2(\mu)}{8} + \phi\frac{\omega}{2} - \alpha\frac{f\cos\theta(\mu)}{2a(\mu)} = F_2(a,\theta). \tag{65}$$

These are the renormalization group equations whose fixed points and stability determine the nature of the solutions. To determine the fixed points, we set

$$\left[\frac{da}{d\mu}\right]_{a_0,\theta_0} = \left[\frac{d\theta}{d\mu}\right]_{a_0,\theta_0} = 0. \tag{66}$$

Here (a_0,θ_0) are the fixed points, of which there are three in number, because using Eq. (66) to combine Eqs. (64) and (65), we obtain a cubic equation for a_0^2.

We are now finished with the four perturbation parameters, and thus we set each of them equal to one. Before doing that we extract some interesting information from their presence in the cubic equation $\lambda^2 a_0^6 - \phi\lambda(8\omega/3)a_0^4 + (16/9)\times(\phi^2\omega^2+\epsilon^2\chi^2)a_0^2 - \alpha^2(16/9)f^2=0$. So that all three roots of this equation are positive, the signs of ϕ and λ must be the same (for the second term). We observe in Eq. (42) that for $\lambda>0$ (the M shaped quartic potential) ϕ is positive, which means that the bend in the amplitude-frequency curve occurs with the value of the external forcing frequency slightly lower than the natural frequency of the oscillator, that is, the curve bends to the right, corresponding to a "soft spring" (see Fig. 1(a)).[8] For $\lambda<0$ ϕ is also negative corresponding to a "hard spring" (see Fig. 1(b)); in this case the curve bends to the left because the bend in the curve occurs when the external frequency is slightly higher than the natural frequency of the oscillator. For $\lambda=0$ or no forcing $(\alpha=0)$, the amplitude equation is no longer a cubic, which is a prerequisite for the jump phenomenon to occur.

The role of damping is a subtle and will be clearer when we do the stability analysis. We now set the perturbation parameters equal to one and obtain

$$a_0^6-\frac{8\omega}{3}a_0^4+\frac{16}{9}(\omega^2+\chi^2)a_0^2-\frac{16}{9}f^2=0. \tag{67}$$

The most relevant consequence[2] of Eq. (67) for the present discussion is that differentiating with respect to ω and then setting $d\omega/da_0=0$ lead to $a_0^2=(8\omega/9)\pm(4/9)\sqrt{\omega^2-3\chi^2}$. Hence, $\omega^2=3\chi^2(a_0^2=8\omega/9=8\sqrt{3}\chi/9)$ corresponds to the critical value of f, above which the tilt in the amplitude-frequency curve begins.

Determining the stability of the roots of the cubic means solving the determinant (all derivatives being taken at $a=a_0$ and $\theta=\theta_0$).

$$\begin{vmatrix}\left(\frac{\partial F_1}{\partial a}\right)_0-\eta & \left(\frac{\partial F_1}{\partial \theta}\right)_0\\ \left(\frac{\partial F_2}{\partial a}\right)_0 & \left(\frac{\partial F_2}{\partial \theta}\right)_0-\eta\end{vmatrix}=\begin{vmatrix}-\frac{\chi}{2}-\eta & \frac{f}{2}\cos\theta_0\\ -\frac{3a_0}{4}-\frac{f\cos\theta_0}{2a_0^2} & \frac{f\sin\theta_0}{2a_0}-\eta\end{vmatrix}=0. \tag{68}$$

We write the determinant as

$$\eta^2-2s\eta+d=0, \tag{69}$$

where

$$s=\frac{1}{4}\left[\frac{f}{a_0}\sin\theta_0-\chi\right], \tag{70}$$

$$d=\frac{1}{4}\left[\frac{f^2}{a_0^2}\cos^2\theta_0-\frac{f\chi}{a_0}\sin\theta_0-\frac{3fa_0}{2}\cos\theta_0\right]. \tag{71}$$

From the condition $[da/d\mu]_{a_0,\theta_0}=0$ given by Eq. (66), we obtain

$$\frac{f}{a_0}\sin\theta_0=-\chi. \tag{72}$$

This result plus Eq. (70) yield the relation

$$s=-\frac{1}{2}\chi<0. \tag{73}$$

Thus in the two roots of Eq. (69) given by $\eta=s\pm\sqrt{s^2-d}$, s is always negative for all three roots. Note that if $s=0$ (no damping), we have divergent solutions (unstable fixed point or unstable spiral, depending on whether $d<0$ or $d>0$) in the long time limit. For $s<0$ the sign and magnitude of d determines the stability of the fixed points,

$$0<d<s^2\quad(\text{stable fixed point}), \tag{74a}$$

$$d<0\quad(\text{unstable fixed point}). \tag{74b}$$

It is cumbersome to solve the cubic equation and then test the stability. The two renormalization group equations offer a hint of a different approach. We use the criteria of Eq. (74b) in the expression for d in Eq. (71) and examine what bounds it implies on θ_0 (and hence on a_0 through Eq. (72)). We write Eq. (71) as

$$4d=\frac{f^2}{a_0^2}\cos^2\theta_0-\frac{f\chi}{a_0}\sin\theta_0-\frac{3fa_0}{2}\cos\theta_0 \tag{75a}$$

$$=\left(\frac{f}{a_0}\cos\theta_0-\frac{3a_0^2}{4}\right)^2+\chi^2-\frac{9a_0^4}{16} \tag{75b}$$

$$=A^2+\chi^2-\frac{9a_0^4}{16}, \tag{75c}$$

with $A=(f/a_0)\cos\theta_0-3a_0^2/4^2$. The inequality $d<s^2$ means that $A^2+\chi^2-(9a_0^4/16)<\chi^2$, that is, $A^2<9a_0^4/16$, yielding

$$f\cos\theta_0\left[\frac{f}{a_0^2}\cos\theta_0-\frac{3a_0}{2}\right]<0. \tag{76}$$

Thus we obtain the first bound on θ_0 as

$$0<\cos\theta_0<\frac{3a_0^3}{2f}. \tag{77}$$

From Eq. (72) we see that $\sin\theta_0$ is always negative, and here we find that $\cos\theta_0$ is always positive. Hence, all three solutions for θ_0 of the cubic are in the fourth quadrant where $\sin\theta_0$ is monotonically increasing. Now let us see what is implied by the other condition for stability $d>0$. From Eq. (75c) we have $A^2>(9a_0^4/16)-\chi^2$ yielding $(f^2/a_0^2)\cos^2\theta_0-(3fa_0/2)\cos\theta_0+\chi^2>0$, which in turn implies

$$\cos\theta_0\le\frac{3a_0^3}{4f}-\frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4}-\frac{4\chi^2}{a_0^2}} \tag{78}$$

or

$$\cos\theta_0\ge\frac{3a_0^3}{4f}+\frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4}-\frac{4\chi^2}{a_0^2}}. \tag{79}$$

Both of these bounds are real because we saw that the bend in the curve starts for $a_0^2\ge8\sqrt{3}\chi/9$, which is greater than $4\chi/3$, thus making the discriminant in the last two equations positive. The bounds are positive as well, with the latter one [Eq. (79)] greater than the former [Eq. (78)].

The stability condition, $0<d<s^2$, confines the value of $\cos\theta_0$ within two separate positive regions,

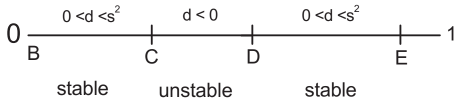

Fig. 2. The stability zones in terms of cos θ_0, as discussed after Eq. (82). Without damping (χ=0), point C merges with point B (see Eq. (80)) as does point D with E (see Eq. (81)), thus making the entire region (BE) unstable.

$$0 < \cos\,\theta_0 < \frac{3a_0^3}{4f} - \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} \tag{80}$$

and

$$\frac{3a_0^3}{4f} + \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} < \cos\,\theta_0 < \frac{3a_0^3}{2f}. \tag{81}$$

Equations (80) and (81) respectively correspond to the regions BC and DE in Fig. 2. The intermediate region, given by region CD and described by the bounds

$$\frac{3a_0^3}{4f} - \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} < \cos\,\theta_0 < \frac{3a_0^3}{4f} + \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}}, \tag{82}$$

corresponds to the condition $d < s^2$ and $d0$, which is equivalent to $d \ngtr 0$ and hence from Eq. (74b) implies instability. In other words, the linear relation between sin θ_0 and a_0 given by Eq. (72) implies that the middle root of a_0 is unstable and the other two are stable. It is clear from Fig. 2 that without damping (χ=0), point C merges with point B (Eq. (80)) as does point D with E (Eq. (81)), thus making the entire region (BE) unstable. Thus, the role played by damping is that it enforces two stable zones (BC and DE) on two sides of an unstable zone (CD), which eventually leads to the jump phenomenon in Fig. 3. From Eq. (82) it is easy to see that the instability zone CD in Fig. 2, given by $(a_0^2/f)\sqrt{(9a_0^2)4)-(4\chi^2/a_0^2)}$, decreases with increasing χ, eventually merging points C and D. This decrease indicates that above a certain value of the damping, the hysteresis bend of the curve in Fig. 3 vanishes completely.[8] We conclude that the jump phenomenon occurs as a consequence of a compromise between the external forcing and the damping.

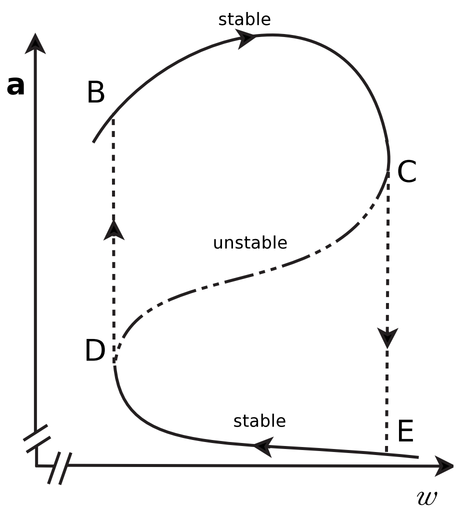

Fig. 3. This figure carries the same information as in Fig. 2. The instability of the middle root of θ_0 implies that the middle root of a_0 is unstable. Damping enforces two stable zones (BC and DE) on two sides of the unstable zone CD, which eventually leads to the jump phenomenon.

IV. CONCLUSION

The renormalization group method is a powerful and beautiful method for doing perturbative solutions to nonlinear differential equations. The freedom to choose initial conditions motivates the use of the renormalization group method in this context. In the secular terms τ divergences are removed by writing $\tau=[\tau-\mu]+\mu$, where μ is an arbitrary time scale. The amplitude and phase terms are renormalized accordingly. The renormalization group equations provide a simple way to analyze the stability of the fixed points of the amplitude equation in comparison to the cumbersome analysis found in the literature.[3,4,8] This method of applying the renormalization group also offers the flexibility of dealing with several perturbation parameters for which a multiple time scale analysis leads to messy calculations.[6]

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