Exact Solutions of a Damped Harmonic Oscillator in a Time Dependent Noncommutative Space

Manjari Dutta[1] · Shreemoyee Ganguly[2] · Sunandan Gangopadhyay[1]

Abstract
In this paper we have obtained the exact eigenstates of a two dimensional damped harmonic oscillator in time dependent noncommutative space. It has been observed that for some specific choices of the damping factor and the time dependent frequency of the oscillator, there exists interesting solutions of the time dependent noncommutative parameters following from the solutions of the Ermakov-Pinney equation. Further, these solutions enable us to get exact analytic forms for the phase which relates the eigenstates of the Hamiltonian with the eigenstates of the Lewis invariant. We then obtain expressions for the matrix elements of the coordinate operators raised to a finite arbitrary power. From these general results we then compute the expectation value of the Hamiltonian. The expectation values of the energy are found to vary with time for different solutions of the Ermakov-Pinney equation corresponding to different choices of the damping factor and the time dependent frequency of the oscillator.

Keywords Noncommutative space · Ermakov-Pinney equation

1 Introduction

The study of time dependent classical as well as quantum harmonic oscillators has appealed to theoretical physicists since time immemorial. In the literature the work by Lewis et al. [1] has lead to an upsurge of analysis of the Hamiltonian for the time dependent quantum

✉ Sunandan Gangopadhyay
sunandan.gangopadhyay@bose.res.in; sunandan.gangopadhyay@gmail.com

Manjari Dutta
manjaridutta@boson.bose.res.in

Shreemoyee Ganguly
ganguly.shreemoyee@gmail.com

1 Department of Theoretical Sciences, S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata 700106, India

2 Department of Basic Science and Humanities, University of Engineering and Management (UEM), B/5, Plot No.III, Action Area-III, Newtown, Kolkata 700156, India

harmonic oscillator using a class of exact invariants designed for such systems [2, 3]. The problem becomes even more fascinating when one has a system of two such oscillators in two-dimensional space. Now, in order to address practical situations one needs to include damping in the system. Although there are several studies on the one-dimensional damped quantum harmonic oscillator in the past [4–8], it's two-dimensional equivalent is a less explored system [9]. The work by Lawson et al. [9] is one of the very few which analyses a two-dimensional damped quantum harmonic oscillator system. The solutions obtained by them for the mentioned system provides a platform to explore the construction of various coherent states with intriguing properties.

In the present work we extend the study by Lawson et al. [9] and consider the two-dimensional damped quantum harmonic oscillator in noncommutative (NC) space. It has been argued that study of quantum mechanical systems in NC space is essential to ensure the attainment of gravitational stability [10] in the present theories of quantum gravity, namely, string theory [11, 12] and loop quantum gravity [13]. The simplest quantum mechanical setting in two dimensional NC space consists of replacing the standard set of commutation relations between the canonical coordinates by NC commutation relations $[X, Y] = i\theta$, where θ is a positive real constant. Quantum mechanical systems in such spaces have been studied extensively in the literature [14–24]. The study of a two-dimensional quantum harmonic oscillator in NC space with time dependent NC parameters was done in [25]. However, their system was an undamped oscillator. The parametrized form of solutions obtained there offered an interesting possibility for study of generalized version of Heisenberg's uncertainty relations. Quantum damped harmonic oscillator on noncommuting two-dimensional space was studied in [26] where the exact propagator of the system was obtained and the thermodynamic properties of the system was investigated using the standard canonical density matrix.

In this work, a two-dimensional damped quantum harmonic oscillator in NC space with time dependent NC parameters is considered once again. We would like to mention that time dependent NC parameters can arise from the renormalization group flow of Newton's gravitational constant G with the energy scale being chosen as the inverse of the cosmic time. The running of Newton's gravitational constant emerges from the solution of an exact functional renormalization group flow equation involving the scale dependent effective action [27–29]. Further, considering time dependent NC parameters also includes the possibility of getting interesting exact solutions of the quantum mechanical system as we shall see subsequently. We would like to stress that our focus of study is different from the work carried out in [26]. We first construct the Hamiltonian and then express it in terms of standard commutative variables. This is done in Section 2. Then we solve the Hamiltonian using the method of invariants [1] and obtain the corresponding eigenfunction in Section 3. In doing so, although we start with the Hamiltonian and corresponding invariant in Cartesian coordinates, eventually we transform our operators to polar coordinates (following closely the procedure suggested in [25]) for ease of solution. The form of the Lewis invariant in Cartesian coordinates with a Zeeman term in the Hamiltonian is an interesting result in itself and it also makes it easier to make a transition to it's polar form. It is to be noted that the eigenfunction of the Hamiltonian is a product of the eigenfunction of the invariant and a phase factor. Both the eigenfunction and phase factor are expressed in terms of time dependent parameters which obey the non-linear differential equation known as Ermakov-Pinney (EP) equation [30, 31]. Next, in Section 4 we judiciously choose the parameters of the damped system such that they satisfy all the equations representing the system as well as provide us with an exact closed form solution of the Hamiltonian. The solutions of the NC parameters obtained in our analysis turns out to be such that the phase factor in an integral form given in [25] is exactly integrable for various kinds of dissipation. Then in Section 5 we **devise** a procedure to calculate the matrix element of a finite arbitrary power of the position operator with respect to the exact solutions for Hamiltonian eigenstates. Using these expressions we proceed to calculate the expectation value of energy and study the evolution of the energy expectation value of the system with time for various types of damping. In Section 6 we summarize our results.

2 Model of the Two-dimensional Harmonic Oscillator

The system we consider is a combination of two non-interacting damped harmonic oscillators in two dimensional NC space. The oscillators have equal time dependent frequencies, time dependent coefficients of friction and equal mass in NC space. Such a model of damped harmonic oscillator was considered in an earlier communication [9] in commutative space. In this work, we extend the model by considering the system in NC space.[1]

The Hamiltonian of the system has the following form,

$$H(t) = \frac{f(t)}{2M}({P_1}^2 + {P_2}^2) + \frac{M\omega^2(t)}{2f(t)}({X_1}^2 + {X_2}^2) \tag{1}$$

where the damping factor $f(t)$ is given by,

$$f(t) = e^{-\int_0^t \eta(s)ds} \tag{2}$$

with $\eta(s)$ being the coefficient of friction. Here $\omega(t)$ is the time dependent angular frequency of the oscillators and M is their mass. It should be noted that in commutative space, the model with $f(t) = e^{-\Gamma t}$ and $\omega(t) = \omega_0$, with Γ and ω_0 being positive constants, is said to be the two-dimensional Caldirola and Kanai Hamiltonian [32, 33]. The position and momentum coordinates (X_i, P_i) are noncommuting variables in NC space, that is, their commutators are $[X_1, X_2] \neq 0$ and $[P_1, P_2] \neq 0$. The corresponding canonical variables (x_i, p_i) in commutative space are such that the commutator $[x_i, p_j] = i\hbar\delta_{i,j}$, $[x_i, x_j] = 0 = [p_i, p_j]$; $(i, j = 1, 2)$.

In order to express the NC Hamiltonian in terms of the standard commutative variables explicitly, we apply the standard Bopp-shift relations [34] ($\hbar = 1$):

$$X_1 = x_1 - \frac{\theta(t)}{2}p_2 \;\; ; \;\; X_2 = x_2 + \frac{\theta(t)}{2}p_1 \tag{3}$$

$$P_1 = p_1 + \frac{\Omega(t)}{2}x_2 \;\; ; \;\; P_2 = p_2 - \frac{\Omega(t)}{2}x_1. \tag{4}$$

Here $\theta(t)$ and $\Omega(t)$ are the NC parameters for space and momentum respectively, such that $[X_1, X_2] = i\theta(t)$, $[P_1, P_2] = i\Omega(t)$ and $[X_1, P_1] = i[1 + \frac{\theta(t)\Omega(t)}{4}] = [X_2, P_2]$; $(X_1 \equiv X$, $X_2 \equiv Y$, $P_1 \equiv P_x$, $P_2 \equiv P_y)$.

The Hamiltonian in terms of (x_i, p_i) coordinates is therefore given by the following relation,

$$H = \frac{a(t)}{2}({p_1}^2 + {p_2}^2) + \frac{b(t)}{2}({x_1}^2 + {x_2}^2) + c(t)(p_1x_2 - p_2x_1). \tag{5}$$

[1] We shall be considering NC phase space in our work. However, we shall generically refer this as NC space.

The time dependent coefficients in the above Hamiltonian are given as,

$$a(t) = \frac{f(t)}{M} + \frac{M\omega^2(t)\theta^2(t)}{4f(t)} \tag{6}$$

$$b(t) = \frac{f(t)\Omega^2(t)}{4M} + \frac{M\omega^2(t)}{f(t)} \tag{7}$$

$$c(t) = \frac{1}{2}\left[\frac{f(t)\Omega(t)}{M} + \frac{M\omega^2(t)\theta(t)}{f(t)}\right]. \tag{8}$$

Here it must be noted that although our Hamiltonian given by (5) has the same form as that in [25] to study a system of a two dimensional harmonic oscillator in NC space, the time dependent Hamiltonian coefficients (given by (8)) have very different form. This is because our system is that of a damped harmonic oscillator in two-dimensional NC space. Thus, the damping factor $f(t)$ modulates and alters the Hamiltonian coefficients from the form considered in earlier study [25].

3 Solution of the Model Hamiltonian

In order to find the solutions of the model Hamiltonian $H(t)$ (5) representing the two-dimensional damped harmonic oscillator in NC space, we follow the route suggested by Lewis et al. [1] in their work. First we construct the time-dependent Hermitian invariant operator $I(t)$ corresponding to our Hamiltonian operator $H(t)$ (given by (5)). This is because if one can solve for the eigenfunctions of $I(t)$, $\phi(x_1, x_2)$, such that,

$$I(t)\phi(x_1, x_2) = \epsilon\phi(x_1, x_2) \tag{9}$$

where ϵ is an eigenvalue of $I(t)$ corresponding to eigenstate $\phi(x_1, x_2)$, one can obtain the eigenstates of $H(t)$, $\psi(x_1, x_2, t)$, using the relation given by Lewis et al. [1] which is as follows,

$$\psi(x_1, x_2, t) = e^{i\Theta(t)}\phi(x_1, x_2) \tag{10}$$

where the real function $\Theta(t)$ which acts as the phase factor will be discussed in details later.

3.1 The Time Dependent Invariant

Next, following the approach taken by Lewis et al. [1], we need to construct the operator $I(t)$ which is an invariant with respect to time, corresponding to the Hamiltonian $H(t)$, as mentioned earlier, such that $I(t)$ satisfies the condition,

$$\frac{dI}{dt} = \partial_t I + \frac{1}{i}[I, H] = 0. \tag{11}$$

The procedure is to choose the Hermitian invariant $I(t)$ to be of the same homogeneous quadratic form defined by Lewis et al. [1] for time-dependent harmonic oscillators. However, since we are dealing with a two-dimensional system in the present study, $I(t)$ takes on the following form,

$$I(t) = \alpha(t)({p_1}^2 + {p_2}^2) + \beta(t)({x_1}^2 + {x_2}^2) + \gamma(t)(x_1 p_1 + p_2 x_2). \tag{12}$$

Here we will consider $\hbar = 1$ since we choose to work in natural units. Now, using the form of $I(t)$ defined by (12) in (11) and equating the coefficients of the canonical variables, we get the following relations,

$$\dot{\alpha}(t) = -a(t)\gamma(t) \tag{13}$$

$$\dot{\beta}(t) = b(t)\gamma(t) \tag{14}$$

$$\dot{\gamma}(t) = 2\left[\,b(t)\alpha(t) - \beta(t)a(t)\,\right] \tag{15}$$

where dot denotes derivative with respect to time t.

To express the above three time dependent parameters α, β and γ in terms of a single time dependent parameter, we parametrize $\alpha(t) = \rho^2(t)$. Substituting this in (13, 15), we get the other two parameters in terms of $\rho(t)$ as,

$$\gamma(t) = -\frac{2\rho\dot{\rho}}{a(t)} \tag{16}$$

$$\beta(t) = \frac{1}{a(t)}\left[\frac{\dot{\rho}^2}{a(t)} + \rho^2 b + \frac{\rho\ddot{\rho}}{a(t)} - \frac{\rho\dot{\rho}\dot{a}}{a^2}\right]. \tag{17}$$

Now, substituting the value of β in (14), we get a non-linear equation in $\rho(t)$ which has the form of the non-linear Ermakov-Pinney (EP) equation with a dissipative term [25, 30, 31]. The form of the non-linear equation is as follows,

$$\ddot{\rho} - \frac{\dot{a}}{a}\dot{\rho} + ab\rho = \xi^2\frac{a^2}{\rho^3}\ . \tag{18}$$

where ξ^2 is a constant of integration. This equation has similar form to the EP equation obtained in [25], which is expected since our $H(t)$ has the same form as theirs. However, once again we should recall the fact that the explicit form of the time-dependent coefficients are different due to the presence of damping.

Now, using the EP equation we get a simpler form of β as,

$$\beta(t) = \frac{1}{a(t)}\left[\frac{\dot{\rho}^2}{a(t)} + \frac{\xi^2 a(t)}{\rho^2}\right]. \tag{19}$$

Next, substituting the expressions of α, β and γ in (12), we get the following expression for $I(t)$,

$$I(t) = \rho^2({p_1}^2 + {p_2}^2) + \left(\frac{\dot{\rho}^2}{a^2} + \frac{\xi^2}{\rho^2}\right)({x_1}^2 + {x_2}^2) - \frac{2\rho\dot{\rho}}{a}(x_1 p_1 + p_2 x_2). \tag{20}$$

The form of the Lewis invariant in Cartesian coordinates will be used later to go over to it's polar coordinate form. The solution of the EP equation under various physically significant conditions shall be discussed later.

3.2 Construction of Ladder Operators

Now that we have the required Hermitian invariant $I(t)$, we proceed to calculate it's eigenstates using the operator approach. For this purpose we need to first construct some ladder operators. To do this, we first need to transform the form of $I(t)$ (given by (20)) to a more manageable form. For this we invoke a unitary transformation using a suitable unitary operator $\hat{U}$ having the following form,

$$\hat{U} = exp\left[-\frac{i\dot{\rho}}{2a(t)\rho}({x_1}^2 + {x_2}^2)\right], \quad \hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \mathbf{I}. \tag{21}$$

Defining,

$$\phi'(x_1, x_2) = \hat{U}\phi(x_1, x_2) \quad , \quad I'(t) = \hat{U} I \hat{U}^\dagger \tag{22}$$

where $\phi(x_1, x_2)$ is an eigenfunction of $I(t)$ as introduced in (9), then, using (9) and (22), we get,

$$I'\phi' = \hat{U} I \hat{U}^\dagger \hat{U}\phi = \hat{U} I\phi = \hat{U}\epsilon\phi = \epsilon\phi'. \tag{23}$$

The transformed expression of the invariant, $I'(t)$, using (22), has the following form,

$$I'(t) = \rho^2({p_1}^2 + {p_2}^2) + \frac{\xi^2}{\rho^2}({x_1}^2 + {x_2}^2) \ . \tag{24}$$

This transformed form of the invariant, $I'(t)$, has exactly the same form as that of the Hamiltonian for a time dependent two-dimensional simple harmonic oscillator. So, we can introduce the corresponding ladder operators for $\hat{I}'(t)$ to be given by,

$$\hat{a}'_j = \frac{1}{\sqrt{2\xi}}\left(\frac{\xi}{\rho}\hat{x}_j + i\rho\hat{p}_j\right) \quad , \quad \hat{a}'^\dagger_j = \frac{1}{\sqrt{2\xi}}\left(\frac{\xi}{\rho}\hat{x}_j - i\rho\hat{p}_j\right) \tag{25}$$

where $j = 1, 2$ and the operators satisfy the commutation relation $[\hat{a}'_i, \hat{a}'^\dagger_j] = \delta_{ij}$.

Now we make the reverse transformation to get the expression of the unprimed ladder operators:

$$\hat{a}_j(t) \ = \ \hat{U}^\dagger \hat{a}'_j \hat{U} = \frac{1}{\sqrt{2\xi}}\left[\frac{\xi}{\rho}x_j + i\rho p_j - \frac{i\dot{\rho}}{a(t)}x_j\right] \tag{26}$$

$$\hat{a_j}^\dagger(t) \ = \ \hat{U}^\dagger \hat{a}'^\dagger_j \hat{U} = \frac{1}{\sqrt{2\xi}}\left[\frac{\xi}{\rho}x_j - i\rho p_j + \frac{i\dot{\rho}}{a(t)}x_j\right]. \tag{27}$$

It can be easily checked using the algebra of the primed ladder operators that $[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij}$.

We now set $\xi = 1$ and consider two linear combinations of the above two operators such that,

$$\hat{a}(t) = -\frac{i}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2) = \frac{1}{2}\left[\rho(\hat{p}_1 - i\hat{p}_2) - \left(\frac{i}{\rho} + \frac{\dot{\rho}}{a(t)}\right)(\hat{x_1} - i\hat{x_2})\right] \tag{28}$$

and

$$\hat{a}^\dagger(t) = \frac{i}{\sqrt{2}}(\hat{a}^\dagger_1 + i\hat{a}^\dagger_2) = \frac{1}{2}\left[\rho(\hat{p}_1 + i\hat{p}_2) + \left(\frac{i}{\rho} - \frac{\dot{\rho}}{a(t)}\right)(\hat{x_1} + i\hat{x_2})\right]. \tag{29}$$

These also satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$.

3.3 Transformation to Polar Coordinates

With the above results in place, we now transform the invariant $I(t)$ and the corresponding ladder operators to polar coordinates for calculational convenience. For this we invoke the transformation of coordinates of the form,[2]

$$x = r cos\theta \quad ; \quad y = r sin\theta \ . \tag{30}$$

[2]The polar angle θ should not be confused with the time dependent NC parameter $\theta(t)$.

The canonical coordinates in polar representation takes the following form,

$$
\begin{aligned}
p_r &= \frac{1}{2}\left(\frac{x_1}{r}p_1 + p_1\frac{x_1}{r} + \frac{x_2}{r}p_2 + p_2\frac{x_2}{r}\right) \\
&= \frac{x_1 p_1 + x_2 p_2}{r} - \frac{i}{2r} \\
&= -i\left(\partial_r + \frac{1}{2r}\right) \qquad (31)
\end{aligned}
$$

$$
p_\theta = (x_1 p_2 - x_2 p_1) = -i\partial_\theta. \qquad (32)
$$

The commutation relations between (r, p_r) and (θ, p_θ) have the form

$$
[r, p_r] = [\theta, p_\theta] = [x_1, p_1] = [x_2, p_2] = i. \qquad (33)
$$

The corresponding anticommutation relation can be found to be,

$$
[r, p_r]_+ = [x_1, p_1]_+ + [x_2, p_2]_+ = 2(x_1 p_1 + p_2 x_2) \qquad (34)
$$

where $[A, B]_+ = AB + BA$ represents anticommutator between operators A, B.

In order to transform the invariant $I(t)$ in polar coordinates, we need to have **a few** other relations which are,

$$
\begin{aligned}
({p_1}^2 + {p_2}^2) &= \left({p_r}^2 + \frac{{p_\theta}^2}{r^2} - \frac{1}{4r^2}\right) \qquad (35) \\
(p_1 + i p_2) &= e^{i\theta}\left[p_r + \frac{i}{r}p_\theta + \frac{i}{2r}\right] \qquad (36) \\
(p_1 - i p_2) &= e^{-i\theta}\left[p_r - \frac{i}{r}p_\theta + \frac{i}{2r}\right]. \qquad (37)
\end{aligned}
$$

Hence the invariant in polar coordinate system is given by,

$$
I(t) = \frac{\xi^2}{\rho^2}r^2 + \left(\rho p_r - \frac{\dot{\rho}}{a}r\right)^2 + \left(\frac{\rho p_\theta}{r}\right)^2 - \left(\frac{\rho\hbar}{2r}\right)^2 \qquad (38)
$$

and the ladder operators in polar coordinate system have the following form,

$$
\begin{aligned}
\hat{a}(t) &= \frac{1}{2}\left[\left(\rho p_r - \frac{\dot{\rho}}{a(t)}r\right) - i\left(\frac{r}{\rho} + \frac{\rho p_\theta}{r} + \frac{\rho}{2r}\right)\right]e^{-i\theta} \\
\hat{a}^\dagger(t) &= \frac{1}{2}e^{i\theta}\left[\left(\rho p_r - \frac{\dot{\rho}}{a(t)}r\right) + i\left(\frac{r}{\rho} + \frac{\rho p_\theta}{r} + \frac{\rho}{2r}\right)\right]. \qquad (39)
\end{aligned}
$$

Now we note from (38) and (39) that both the invariant $I(t)$ and the ladder operators have the same form as those used in [25] to study the undamped harmonic oscillator in NC space. The time-dependent coefficients involved in the present study however differ due to the damping present in our system. Thus, we can just borrow the expression of eigenfunction and the phase factors from [25] for our present system.

3.4 Eigenfunction and Phase Factor

We depict the set of eigenstates of the invariant operator $I(t)$ as $|n, l\rangle$, following the convention in [25]. Here, n and l are integers such that $n+l \geqslant 0$. So we have the condition $l \geqslant -n$.

Thus, if $l = -n + m$, then m is a positive integer; and the corresponding eigenfunction in polar coordinate system has the following form (restoring $\hbar$),

$$\phi_{n,m-n}(r,\theta) = \langle r,\theta|n, m-n\rangle \tag{40}$$

$$= \lambda_n \frac{(i\sqrt{\hbar}\rho)^m}{\sqrt{m!}} r^{n-m} e^{i(m-n)\theta - \frac{a(t)-i\rho\dot{\rho}}{2a(t)\hbar\rho^2}r^2} U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right) \tag{41}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi n!(\hbar\rho^2)^{1+n}} . \tag{42}$$

Here, $U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right)$ is Tricomi's confluent hypergeometric function [35, 36] and the eigenfunction $\phi_{n,m-n}(r,\theta)$ satisfies the following orthonormality relation,

$$\int_0^{2\pi} d\theta \int_0^\infty r dr \phi^*_{n,m-n}(r,\theta)\phi_{n',m'-n'}(r,\theta) = \delta_{nn'}\delta_{mm'}. \tag{43}$$

Again following [25], the expression of the phase factor $\Theta(t)$ is given by,

$$\Theta_{n,l}(t) = (n+l)\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right) dT . \tag{44}$$

For a given value of $l = -n + m$, it would be given by [25],

$$\Theta_{n,m-n}(t) = m\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right) dT . \tag{45}$$

We shall use this expression to compute the phase explicitly as a function of time for various physical cases in the subsequent discussion.

The eigenfunction of the Hamiltonian therefore reads (using (10), (41) and (45))

$$\psi_{n,m-n}(r,\theta,t) = e^{i\Theta_{n,m-n}(t)}\phi_{n,m-n}(r,\theta)$$

$$= \lambda_n \frac{(i\sqrt{\hbar}\rho)^m}{\sqrt{m!}} \exp\left[im\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right) dT\right]$$

$$\times r^{n-m} e^{i(m-n)\theta - \frac{a(t)-i\rho\dot{\rho}}{2a(t)\hbar\rho^2}r^2} U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right). \tag{46}$$

4 Solutions for the Noncommutative Damped Oscillator

In this paper we are primarily interested in damped oscillators in NC space. For this purpose we want to find the eigenfunctions of the corresponding Hamiltonian under various types of damping. The various kinds of damping are represented by various forms of the time dependent coefficients of the Hamiltonian, namely, $a(t)$, $b(t)$ and $c(t)$. However, the various forms must be constructed in such a way that they satisfy the non-linear EP equation given by (18). The procedure of this construction of exact analytical solutions is based on the Chiellini integrability condition [37] and this formalism was followed in [25]. We shall do the same in this paper. So, for various forms of $a(t)$ and $b(t)$, we get the corresponding form of $\rho(t)$ using the EP equation together with the Chiellini integrability condition. In other words, the set of values of $a(t)$, $b(t)$ and $\rho(t)$ that we use must be a solution set of the EP equation consistent with the Chiellini integrability condition. In the subsequent discussion we shall proceed to obtain solutions of the EP equation for the damped NC oscillator.

4.1 Solution Set-I for Ermakov-Pinney Equation : Exponentially Decaying Solutions

4.1.1 The Solution Set

The simplest kind of solution set of EP equation under damping is the exponentially decaying set used in [25]. The solution set is given by the following relations,

$$a(t) = \sigma e^{-\vartheta t} \quad , \quad b(t) = \Delta e^{\vartheta t} \quad , \quad \rho(t) = \mu e^{-\vartheta t/2} \tag{47}$$

where σ, Δ and μ are constants. Here, ϑ is any positive real number. Substituting the expression of $a(t)$, $b(t)$ and $\rho(t)$ in the EP equation, we can easily verify the relation between these constants to be as follows,

$$\mu^4 = \frac{\xi^2\sigma^2}{\sigma\Delta - \frac{1}{4}\vartheta^2} \; . \tag{48}$$

4.1.2 Study of the Corresponding Eigenfunctions

We now write down the eigenfunctions of the Hamiltonian for the choosen set of time-dependent coefficients. For this endeavour we need to choose explicit forms of the damping factor $f(t)$ and angular frequency of the oscillator $\omega(t)$. The eigenfunction of the invariant $I(t)$ (which is given by (41)) takes on the following form for the solution set-I:

$$\phi_{n,m-n}(r,\theta) = \lambda_n \frac{(i\mu e^{-\vartheta t/2})^m}{\sqrt{m!}} r^{n-m} e^{i(m-n)\theta - \frac{2\sigma + i\mu^2\vartheta}{4\sigma\mu^2 e^{-\vartheta t}} r^2} U\left(-m, 1-m+n, \frac{r^2 e^{\vartheta t}}{\mu^2}\right) \tag{49}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi \, n! \, [\mu^2 \, exp\,(-\vartheta t)]^{1+n}} \; . \tag{50}$$

In order to obtain explicit expressions of the phase factors for various cases of the damping factor, we choose both the functions $\omega(t)$ and $\eta(t)$ as follows.

⟨*A*⟩ Solution Set-Ia Firstly, we choose the damping factor $f(t) = 1$. Thus, in this case the damping in the system is due to the exponentially decaying frequency $\omega(t)$. For this purpose we set,

$$\eta(t) = 0 \;\Rightarrow\; f(t) = 1 \tag{51}$$

$$\omega(t) = \omega_0 \, exp(-\Gamma t/2) \; . \tag{52}$$

Substituting the expressions for $a(t)$, $b(t)$, $\omega(t)$ and $f(t)$ in the (6) and (7), we get the time dependent NC parameters as,

$$\theta(t) = \frac{2}{M\omega_0} exp\,[\Gamma t/2]\sqrt{M\sigma\, exp(-\vartheta t) - 1} \tag{53}$$

$$\Omega(t) = 2\sqrt{M[\Delta\, exp\,(\vartheta t) - M\omega_0^2\, exp\,(-\Gamma t)]}. \tag{54}$$

It can be checked that in the limit $\Gamma \to 0$, that is, for constant frequency, the expressions for $\theta(t)$ and $\Omega(t)$ reduce to those in [25]. When $\vartheta = \Gamma$, then the solutions take the form,

$$\theta(t) = \frac{2}{M\omega_0}\sqrt{M\sigma - e^{\Gamma t}} \tag{55}$$

$$\Omega(t) = 2\sqrt{M[\Delta\, exp\,(\Gamma t) - M\omega_0^2\, exp\,(-\Gamma t)]}. \tag{56}$$

Substituting these relations in the expression for $c(t)$ in (8), we get an expression for the phase in a closed form as,

$$c(t) = \sqrt{\frac{\Delta\, exp\,(\Gamma t) - M{\omega_0}^2\, exp\,(-\Gamma t)}{M}} + \omega_0\, exp\,(-\Gamma t/2)\sqrt{M\sigma\, exp\,(-\Gamma t) - 1}\,. \quad (57)$$

Substituting the expressions of $a(t)$, $\rho(t)$ and $c(t)$ in (45), we get,

$$\begin{aligned}\Theta_{n,l}(t) = (n+l)\frac{\omega_0}{2\sqrt{M\sigma}\Gamma}&\left[log_e\frac{e^{\Gamma t} - 2M\sigma - 2\sqrt{M\sigma(M\sigma - e^{\Gamma t})}}{1 - 2M\sigma - 2\sqrt{M\sigma(M\sigma-1)}}\right.\\ &\left.-\Gamma t - 2\sqrt{M\sigma(M\sigma e^{-2\Gamma t} - e^{-\Gamma t})} + 2\sqrt{M\sigma(M\sigma-1)}\right]\\ &+\frac{2(n+l)}{\Gamma}\left[\sqrt{\frac{\Delta}{M}e^{\Gamma t} - \omega_0^2 e^{-\Gamma t}} - \sqrt{\frac{\Delta}{M} - \omega_0^2}\right.\\ &\left.+2i\omega_0\left\{e^{-\Gamma t/2}{}_2F_1\left(-\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{\Delta e^{2\Gamma t}}{M\omega_0^2}\right) - {}_2F_1\left(-\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{\Delta}{M\omega_0^2}\right)\right\}\right]\\ &-\frac{\sigma}{\mu^2}(n+l)t \end{aligned} \quad (58)$$

where ${}_2F_1(a,b,c;t)$ is said to be the Gauss hypergeometric function. It is interesting to note that the solutions of the time dependent NC parameters enable us to get an exact analytic expression for the phase factor. It is further interesting to observe that the phase has a complex part which indicates that the wave function decays with time.

⟨*B*⟩ Solution Set-Ib Here the oscillator is damped due to the damping factor $f(t)$ and the frequency $\omega(t)$ is a constant. This situation can be depicted by the following relations,

$$f(t) = exp\,(-\Gamma t)\,;\ \ \omega(t) = \omega_0. \quad (59)$$

Substituting these relations in (6) and (7), we get the time dependent NC parameters as,

$$\theta(t) = \frac{2}{M\omega_0}\sqrt{M\sigma\, exp\,(-\vartheta t) - exp\,(-\Gamma t)}\; e^{-\Gamma t/2} \quad (60)$$

$$\Omega(t) = 2e^{\Gamma t}\sqrt{M\,[\Delta\, exp\,(\vartheta - \Gamma)t - M{\omega_0}^2]}. \quad (61)$$

It can be checked that in the limit $\Gamma \to 0$, that is, for constant frequency, the expressions for $\theta(t)$ and $\Omega(t)$ reduce to those in [25]. When $\vartheta = \Gamma$, then the solutions take the form,

$$\theta(t) = \frac{2}{M\omega_0}\sqrt{M\sigma - 1}\; e^{-\Gamma t} \quad (62)$$

$$\Omega(t) = 2e^{\Gamma t}\sqrt{M\,[\Delta - M{\omega_0}^2]}. \quad (63)$$

Substituting these relations in the expression for $c(t)$ in (8), we get,

$$c(t) = \sqrt{\frac{\Delta - M{\omega_0}^2}{M}} + \omega_0\sqrt{M\sigma - 1} = constant\,. \quad (64)$$

Substituting the expressions of $a(t)$, $\rho(t)$ and $c(t)$ in (45) , we get an expression for the phase in a closed form as,

$$\Theta_{n,l}(t) = (n+l)\left[-\frac{\sigma}{\mu^2} + \sqrt{\frac{\Delta - M{\omega_0}^2}{M}} + \omega_0\sqrt{M\sigma - 1}\right] t\,. \tag{65}$$

Once again we are able to obtain an exact expression for the phase, in this case varying linearly with time. It is important to note that whether the phase is real in this case depends crucially on the parameters Δ, M, σ, ω_0. The phase $\Theta_{n,l}$ is real if $\Delta - M\omega_0^2 \geq 0$ and $M\sigma \geq 1$, else it is complex.

⟨*C*⟩ Solution Set-Ic Here the oscillator is damped due to the damping factor $f(t)$ and the time-dependent frequency $\omega(t)$; both of which are exponentially decaying. Thus, we set,

$$f(t) = exp\,(-\Gamma t)\ ;\ \ \omega(t) = \omega_0\, exp\,(-\Gamma t/2). \tag{66}$$

Substituting these relations in (6) and (7), we get the time dependent NC parameters to be,

$$\theta(t) = \frac{2}{M\omega_0}\sqrt{(M\sigma e^{-(\vartheta-\Gamma)t} - 1)}e^{-\Gamma t/2} \tag{67}$$

$$\Omega(t) = 2\sqrt{M\,[\Delta\, exp(\vartheta t) - M{\omega_0}^2\ ]}\ e^{\Gamma t/2}. \tag{68}$$

It can be checked that in the limit $\Gamma \to 0$, that is, for constant frequency, the expressions for $\theta(t)$ and $\Omega(t)$ reduce to those in [25]. When $\vartheta = \Gamma$, then the solutions take the form,

$$\theta(t) = \frac{2}{M\omega_0}\sqrt{(M\sigma - 1)}e^{-\Gamma t/2} \tag{69}$$

$$\Omega(t) = 2\sqrt{M\,[\Delta\, exp(\Gamma t) - M{\omega_0}^2\ ]}\ e^{\Gamma t/2}. \tag{70}$$

Substituting these relations in the expression for $c(t)$ in (8), we get,

$$c(t) = \sqrt{\frac{\Delta - M{\omega_0}^2\ \exp[-\Gamma t]}{M}} + \omega_0\, e^{-\Gamma t/2}\sqrt{M\sigma - 1}\,. \tag{71}$$

Substituting the expressions of $a(t)$, $\rho(t)$ and $c(t)$ in (45), we obtain an expression for the phase in a closed form as,

$$\begin{aligned}\Theta_{n,l}(t) = \frac{(n+l)}{\Gamma\sqrt{M}}\Bigg[&\sqrt{\Delta}\,\Gamma\, t + 2\sqrt{\Delta - M\omega_0^2} - 2\sqrt{\Delta - M\omega_0^2 \exp(-\Gamma t)}\\ &+ 2\sqrt{\Delta}\, log\left(\frac{\Delta + \sqrt{\Delta[\Delta - M\omega_0^2\, \exp(-\Gamma t)]}}{\Delta + \sqrt{\Delta[\Delta - M\omega_0^2]}}\right)\Bigg]\\ &- (n+l)\left[\frac{\sigma\, t}{\mu^2} + \frac{2}{\Gamma}\,\omega_0\left(e^{-\Gamma t/2} - 1\right)\sqrt{M\sigma - 1}\right].\end{aligned} \tag{72}$$

4.2 Solution Set-II for Ermakov-Pinney Equation: Rationally Decaying Solutions

4.2.1 The Solution Set

We now consider rationally decaying solutions of the EP equation similar to that used in [25] which is of the form,

$$
\begin{aligned}
a(t) &= \frac{\sigma\left(1+\frac{2}{k}\right)^{(k+2)/k}}{(\Gamma t+\chi)^{(k+2)/k}} \\
b(t) &= \frac{\Delta\left(\frac{k}{k+2}\right)^{(2-k)/k}}{(\Gamma t+\chi)^{(k-2)/k}} \quad \Rightarrow \quad \frac{\Delta\left(1+\frac{2}{k}\right)^{(k-2)/k}}{(\Gamma t+\chi)^{(k-2)/k}} \\
\rho(t) &= \frac{\mu\left(1+\frac{2}{k}\right)^{1/k}}{(\Gamma t+\chi)^{1/k}}
\end{aligned}
\tag{73}
$$

where σ, Δ, μ, Γ and χ are constants such that $(\Gamma t+\chi) \neq 0$, and k is an integer. Substituting the expressions of $a(t)$, $b(t)$, and $\rho(t)$ in the EP equation, we can easily verify the relation between these constants to be as follows,

$$
\Gamma^2\mu = (k+2)^2\,(\sigma\Delta\mu - \frac{\xi^2\sigma^2}{\mu^3}). \tag{74}
$$

4.2.2 Study of the Corresponding Eigenfunctions

The eigenfunction of the invariant operator $I(t)$ (given by (41)) for this solution Set-II is given by,

$$
\begin{aligned}
\phi_{n\,,\,m-n}(r,\theta) \;=\; & \lambda_n \frac{(i\mu)^m}{\sqrt{m!}}\left[\frac{k+2}{k(\Gamma t+\chi)}\right]^{m/k} r^{n-m} e^{i\theta(m-n) - \frac{[\sigma\,(k+2)+i\mu^2\Gamma]\,(\Gamma t+\chi)^{2/k}\,k^{2/k}}{2\sigma\,(k+2)^{(k+2)/k}\mu^2} r^2} \\
& \times\; U\left(-m, 1-m+n, \frac{r^2[k(\Gamma t+\chi)]^{2/k}}{\mu^2\,(k+2)^{2/k}}\right)
\end{aligned}
\tag{75}
$$

where λ_n is given by

$$
\lambda_n^2 = \frac{1}{\pi\; n!\mu^{2n+2}}\left[\frac{k(\Gamma t+\chi)}{k+2}\right]^{2(1+n)/k}. \tag{76}
$$

In order to get the eigenfunction of the Hamiltonian $H(t)$, we need to calculate the associated phase factor. Once again for this we need to fix up the forms of the damping factor $f(t)$ and angular frequency $\omega(t)$ of the oscillator. In order to explore the solution of $H(t)$ for rationally decaying coefficients, we choose a rationally decaying form for $\omega(t)$ and set $f(t) = 1$. Thus, we have the following relations,

$$
\eta(t) \;=\; 0 \;\Rightarrow\; f(t) = 1 \tag{77}
$$

$$
\omega(t) \;=\; \frac{\omega_0}{(\Gamma\, t+\chi)}\,. \tag{78}
$$

Substituting these relations in (6) and (7), we get the time dependent NC parameters as,

$$\theta(t) = \frac{2\,(\Gamma\,t+\chi)}{M\,\omega_0}\sqrt{M\sigma\left[\frac{(k+2)}{k\,(\Gamma t\,+\,\chi)}\right]^{(k+2)/k} - 1} \tag{79}$$

$$\Omega(t) = 2\sqrt{M\Delta\left[\frac{k+2}{k(\Gamma t+\chi)}\right]^{(k-2)/k} - \frac{M^2\omega_0^2}{(\Gamma\,t+\chi)^2}}\,. \tag{80}$$

We now consider $k = 2$. This enables us to integrate the expression for the phase factor (given by (45)). The simplified forms of $a(t)$, $b(t)$ and $\rho(t)$ for $k = 2$ read,

$$a(t) = \frac{4\sigma}{(\Gamma t+\chi)^2}\ ,\ b(t) = \Delta\ ,\ \rho(t) = \left[\frac{2\mu^2}{\Gamma t+\chi}\right]^{1/2}. \tag{81}$$

Substituting these relations in the expression for $c(t)$ in (8) gives,

$$c(t) = \frac{\omega_0}{(\Gamma\,t+\chi)}\sqrt{\frac{4\sigma\,M}{(\Gamma t+\chi)^2} - 1} + \sqrt{\frac{\Delta}{M} - \frac{\omega_0^2}{(\Gamma\,t+\chi)^2}}\ . \tag{82}$$

Substituting these expressions for $a(t)$, $\rho(t)$ and $c(t)$ for $k = 2$ in (45), we get the following expression for the phase factor in a closed form as,

$$\begin{aligned}\Theta_{n,l}(t) = \frac{(n+l)}{\Gamma}\Bigg[&\omega_0\ tan^{-1}\left(\frac{\omega_0}{\sqrt{\frac{\Delta}{M}(\Gamma\,t+\chi)^2-\omega_0^2}}\right) + \sqrt{\frac{\Delta\,(\Gamma\,t+\chi)^2}{M} - \omega_0^2}\\ &-\frac{2\sigma}{\mu^2}\,log_e\,\frac{(\chi+\Gamma\,t)}{\chi} - \sqrt{\frac{\Delta}{M}\chi^2-\omega_0^2} - \omega_0\ tan^{-1}\left(\frac{\omega_0}{\sqrt{\frac{\Delta}{M}\chi^2-\omega_0^2}}\right)\Bigg]\\ &+\frac{\omega_0(n+l)}{\Gamma}\left[\frac{\sqrt{4\,\sigma\,M-\chi^2}}{\chi} - \frac{\sqrt{4\,\sigma\,M-(\chi+\Gamma\,t)^2}}{(\chi+\Gamma\,t)}\right.\\ &\left.+i\,log_e\frac{(\chi+\Gamma\,t)+\sqrt{(\chi+\Gamma\,t)^2-4\,\sigma\,M}}{\chi+\sqrt{\chi^2-4\,\sigma\,M}}\right].\end{aligned} \tag{83}$$

We can now get the eigenfunction of this rationally decaying damped system using (10).

4.3 Solution Set-III for Ermakov-Pinney equation: Elementary Solution

4.3.1 The Solution Set

We now propose a simple method of obtaining a solution of the EP equation. The method is as follows. Choosing $\rho(t)$ to be any arbitrary time dependent function and taking it's time derivative as proportional to $a(t)$, that is, $a(t) = constant \times \dot{\rho}$ and setting $b(t) = constant \times \frac{a}{\rho^4}$, we observe that these would always satisfy the EP equation along with a certain constraint relation among the constants.

Here we consider a simple solution which is a special case of the above solution for the EP equation. We call this the elementary solution which reads,

$$a(t) = \sigma\ \ ,\ b(t) = \frac{\Delta}{(\Gamma\,t\,+\,\chi)^4}\ ,\ \rho(t) = \mu(\Gamma t\,+\,\chi) \tag{84}$$

where Γ, χ, μ, σ and Δ are constants. The above solution set satisfy the EP equation with the following constraint relation,

$$\Delta\mu^4 = \xi^2\sigma \ . \tag{85}$$

4.3.2 Study of the Corresponding Eigenfunctions

The eigenfunctions of the invariant operator $I(t)$ for this solution set is given by,

$$\phi_{n,m-n}(r,\theta) = \lambda_n \frac{[i\mu(\Gamma t+\chi)]^m}{\sqrt{m!}} r^{n-m} e^{i\theta(m-n)-\frac{\sigma-i\mu^2\Gamma(\Gamma t+\chi)}{2\sigma\mu^2(\Gamma t+\chi)^2}r^2} \times U\left(-m, 1-m+n, \frac{r^2}{\mu^2(\Gamma t+\chi)^2}\right) \tag{86}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi n![\mu(\Gamma t+\chi)]^{2+2n}} \ . \tag{87}$$

In order to get an eigenfunction of the Hamiltonian, we calculate the phase factor for a particular case of the damped harmonic oscillator where the angular frequency $\omega(t)$ is rationally decaying and the damping factor $f(t)$=1. Thus, we set,

$$\eta(t) = 0 \ \Rightarrow \ f(t) = 1 \tag{88}$$

$$\omega(t) = \frac{\omega_0}{(\Gamma t+\chi)} \tag{89}$$

where Γ and χ are real constants. Substituting these relations in (6) and (7), we get the time dependent NC parameters as,

$$\theta(t) = \frac{2(\Gamma t+\chi)}{\omega_0 M}\sqrt{M\sigma - 1} \tag{90}$$

$$\Omega(t) = 2\sqrt{\frac{M\Delta}{(\Gamma t+\chi)^4} - \frac{M^2\omega_0^2}{(\Gamma t+\chi)^2}} \ . \tag{91}$$

Substituting these relations in the expression for $c(t)$ in (8), we get,

$$c(t) = \sqrt{\frac{\Delta}{M(\Gamma t+\chi)^4} - \frac{\omega_0^2}{(\Gamma t+\chi)^2}} + \frac{\omega_0}{(\Gamma t+\chi)}\sqrt{M\sigma - 1}. \tag{92}$$

Substituting these expressions of $a(t)$, $\rho(t)$ and $c(t)$ in (45), we obtain an expression for the phase factor in a closed form as,

$$\Theta_{n,l}(t) = (n+l)\left[\omega_0\frac{\sqrt{M\sigma-1}}{\Gamma}\log\frac{(\Gamma t+\chi)}{\chi} - \frac{\sigma t}{\mu^2\chi(\Gamma t+\chi)}\right] + \frac{(n+l)}{\Gamma}\left[\sqrt{\frac{\Delta}{M\chi^2} - \omega_0^2} - \sqrt{\frac{\Delta}{M(\Gamma t+\chi)^2} - \omega_0^2} + \omega_0\left\{\tan^{-1}\left(\frac{\omega_0\chi}{\sqrt{\frac{\Delta}{M} - \chi^2\omega_0^2}}\right) - \tan^{-1}\left(\frac{\omega_0(\Gamma t+\chi)}{\sqrt{\frac{\Delta}{M} - \omega_0^2(\Gamma t+\chi)^2}}\right)\right\}\right] . \tag{93}$$

We can now get the eigenfunction of this system by using (10).

5 Expectation Values

In this section, we intend to calculate the expectation value of energy. For this we need to calculate the expectation value of the Hamiltonian $H(t)$ in it's own eigenstates. The expectation value $\langle H\rangle$ is given by (using (5)),

$$\langle H\rangle = \frac{a(t)}{2}(\langle {p_1}^2\rangle + \langle {p_2}^2\rangle) + \frac{b(t)}{2}(\langle {x_1}^2\rangle + \langle {x_2}^2\rangle) + c(t)(\langle p_1 x_2\rangle - \langle p_2 x_1\rangle) \ . \quad (94)$$

To calculate this we need to get the expectation value of the individual canonical operators. To set up our notation we denote the eigenstates of the Hamiltonian $H(t)$ by $|n, l\rangle_H$.

5.1 Matrix Elements of the Coordinate Operators Raised to Arbitrary Finite Powers

We start by calculating the matrix element of an arbitrary power of x, ${}_H\langle n, l|x^k|n, l\rangle_H$, which is given by

$$\begin{aligned} {}_H\langle n, m-n|x^k|n, m'-n\rangle_H &= \int r dr d\theta \ {}_H\langle n, m-n|r, \theta\rangle\langle r, \theta|r^k cos^k\theta|n, m'-n\rangle_H \\ &= \frac{1}{2^k} e^{i(\Theta_{n,m'-n} - \Theta_{n,m-n})} \int r^{k+1} dr d\theta \ (e^{i\theta} + e^{-i\theta})^k \\ &\quad \times \phi^*_{n,m-n}(r,\theta)\phi_{n,m'-n}(r,\theta) \end{aligned} \quad (95)$$

where we have used the relations, $|n, l\rangle_H = e^{i\Theta_{n,l}}|n, l\rangle$ where $|n, l\rangle_H$ and $|n, l\rangle$ are eigenstates of the Hamiltonian $H(t)$ and Lewis invariant $I(t)$ respectively. We have also used the relation $\langle r, \theta|n, m'-n\rangle = \phi_{n,m'-n}(r, \theta)$, with ϕ being the eigenfunction of $I(t)$. Now, (95) can be rewritten as,

$$\begin{aligned} {}_H\langle n, m-n|x^k|n, m'-n\rangle_H &= \frac{\pi}{2^{k-1}} \sum_{r=0}^{k} {}^kC_r \delta_{m',m+2r-k} A(n, m, m+2r-k) \\ &\quad \times \int_0^\infty r \ dr \ r^{2(n-m-r+k)} e^{\frac{-r^2}{\hbar\rho^2}} \\ &\quad \times U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right) \\ &\quad \times U\left(-m-2r+k, 1-m-2r+k+n, \frac{r^2}{\hbar\rho^2}\right) \end{aligned} \quad (96)$$

where $A(n, m, m+2r-k) = e^{i(\Theta_{n,m-n+2r-k} - \Theta_{n,m-n})} \lambda_n^2 \frac{(-i\hbar^{1/2}\rho)^m (i\hbar^{1/2}\rho)^{m+2r-k}}{\sqrt{m!(m+2r-k)!}}$.

Now defining $w = -\frac{r^2}{\hbar\rho^2}$, we have,

$$\begin{aligned} {}_H\langle n, m-n|x^k|n, m'-n\rangle_H = \sum_{r=0}^{k} \frac{\pi}{2^k} e^{i(\Theta_{n,m-n+2r-k} - \Theta_{n,m-n})} (-1)^{k+r} i^{-k} \ {}^kC_r \delta_{m',m+2r-k} \\ \times \lambda_n^2 (\hbar^{1/2}\rho)^{2n+k+2} \sqrt{m!(m+2r-k)!} \\ \times \int_0^\infty dw w^{n-m-r+k} e^{-w} L_m^{(n-m)}(w) L_{m+2r-k}^{(n-m-2r+k)}(w) \end{aligned} \quad (97)$$

where we have used the following result on special functions [35, 36],

$$L_n^{(\zeta)}(w) = \frac{(-1)^n}{n!} U(-n, \zeta+1, w) \quad (98)$$

where $L_n^{(\zeta)}(w)$ are associated Laguerre polynomials.

Now, we get using the relation for phase given in [25],

$$\Theta_{n,l} = (n+l)\int^t \left[c(\tau) - \frac{a(\tau)}{\rho^2(\tau)}\right] d\tau \tag{99}$$

the following relation,

$$\begin{aligned} e^{i(\Theta_{n,m-n+2r-k}-\Theta_{n,m-n})} &= e^{i[\{(n+m-n+2r-k)-(n+m-n)\}\int^t (c(\tau)-\frac{a(\tau)}{\rho^2(\tau)})d\tau]} \\ &= e^{i(0+2r-k)\int^t (c(\tau)-\frac{a(\tau)}{\rho^2(\tau)})d\tau} \\ &= e^{i\Theta_{0,2r-k}}. \end{aligned} \tag{100}$$

So, we finally get the following relation for the matrix element of x^k,

$$\begin{aligned} {}_H\langle n, m-n|x^k|n, m'-n\rangle_H &= \sum_{r=0}^{k} \frac{\pi}{2^k} e^{i\Theta_{0,2r-k}} (-1)^{k+r} i^{-k}\, {}^kC_r \delta_{m',m+2r-k} \\ &\times \lambda_n^2 (\hbar^{1/2}\rho)^{2n+k+2} \sqrt{m!(m+2r-k)!} \\ &\times \int_0^\infty dw\, w^{n-m-r+k} e^{-w} L_m^{(n-m)}(w) L_{m+2r-k}^{(n-m-2r+k)}(w). \end{aligned} \tag{101}$$

This is a new result in this paper and can be used to obtain the matrix element or expectation value of any power of x. For the sake of completeness, we also write down the matrix element of x^k in the eigenstates of the Lewis invariant $I(t)$, which reads

$$\begin{aligned} \langle n, m-n|x^k|n, m'-n\rangle &= \sum_{r=0}^{k} \frac{\pi}{2^k} (-1)^{k+r} i^{-k}\, {}^kC_r \delta_{m',m+2r-k} \\ &\times \lambda_n^2 (\hbar^{1/2}\rho)^{2n+k+2} \sqrt{m!(m+2r-k)!} \\ &\times \int_0^\infty dw\, w^{n-m-r+k} e^{-w} L_m^{(n-m)}(w) L_{m+2r-k}^{(n-m-2r+k)}(w). \end{aligned} \tag{102}$$

Note that the phase factor does not appear in the above result.

Now, we proceed to evaluate the matrix element ${}_H\langle n, m-n|x|n, m'-n\rangle_H$ using the expression obtained in (101). This reads

$$\begin{aligned} {}_H\langle n, m-n|x|n, m'-n\rangle_H =& {}_H\langle n, m-n|x^k\ |_{k=1;\, r=0}\ |n, m'-n\rangle_H \\ &+ {}_H\langle n, m-n|x^k\ |_{k=1;\, r=1}\ |n, m'-n\rangle_H. \end{aligned} \tag{103}$$

Evaluating the above matrix elements give,

$${}_H\langle n, m-n|x^k\ |_{k=1;\, r=0}\ |n, m'-n\rangle_H = -\frac{i}{2}(\rho\hbar^{1/2})\sqrt{m}e^{-i\Theta_{0,1}}\delta_{m,m'+1} \tag{104}$$

$${}_H\langle n, m-n|x^k\ |_{k=1;\, r=1}\ |n, m'-n\rangle_H = \frac{i}{2}(\rho\hbar^{1/2})\sqrt{m'}e^{i\Theta_{0,1}}\delta_{m',m+1}. \tag{105}$$

In order to obtain (104) and (105), we used the following relations involving the associated Laguerre polynomials,

$$\begin{aligned} L_n^{(\zeta)}(w) &= L_n^{(\zeta+1)}(w) - L_{n-1}^{(\zeta+1)}(w) \\ \int_0^\infty dw\, w^\zeta e^{-w} L_n^{(\zeta)}(w) L_m^{\zeta}(w) &= \frac{(n+\zeta)!}{n!}\delta_{n,m}\,. \end{aligned} \tag{106}$$

Combining (104) and (105), we get the following expression,

$$_H\langle n, m-n|x|n, m'-n\rangle_H = \frac{i}{2}(\rho\hbar^{1/2})[\sqrt{m'}e^{i\Theta_{0,1}}\delta_{m',m+1} - \sqrt{m}e^{-i\Theta_{0,1}}\delta_{m,m'+1}]. \quad (107)$$

Next, we evaluate,

$$_H\langle n, m-n|x^2|n, m'-n\rangle_H = _H\langle n, m-n|x^k|_{k=2;\, r=0}|n, m'-n\rangle_H$$
$$+_H\langle n, m-n|x^k|_{k=2;\, r=1}|n, m'-n\rangle_H +_H\langle n, m-n|x^k|_{k=2;\, r=2}|n, m'-n\rangle_H. \quad (108)$$

Evaluation of the above matrix elements yield,

$$_H\langle n, m-n|x^k|_{k=2;\, r=0}|n, m'-n\rangle_H = -\frac{1}{4}(\hbar\rho^2)e^{-i\Theta_{0,2}}\delta_{m',m-2}\sqrt{m(m-1)}$$
$$_H\langle n, m-n|x^k|_{k=2;\, r=1}|n, m'-n\rangle_H = \frac{1}{2}(\hbar\rho^2)e^{-i\Theta_{0,0}}\delta_{m,m'}(m+n+1)$$
$$_H\langle n, m-n|x^k|_{k=2;\, r=2}|n, m'-n\rangle_H = -\frac{1}{4}(\hbar\rho^2)e^{i\Theta_{0,2}}\delta_{m',m+2}\sqrt{(m+2)(m+1)}. \quad (109)$$

In order to calculate the above expressions, apart from the relations between special functions given by (106), we need the following relation,

$$\int_0^\infty dw\; w^{k+p}e^{-w}L_n^k(w)L_n^k(w) = \frac{(n+k)!}{n!} \times (2n+k+1)^p\,. \quad (110)$$

So we have,

$$_H\langle n, m-n|x^2|n, m'-n\rangle_H = \frac{(\hbar\rho^2)}{2}\delta_{m,m'}(m+n+1)$$
$$-\frac{(\hbar\rho^2)}{4}\Big[e^{-i\Theta_{0,2}}\delta_{m',m-2}\sqrt{m(m-1)}$$
$$+e^{i\Theta_{0,2}}\delta_{m',m+2}\sqrt{(m+2)(m+1)}\Big]. \quad (111)$$

It is to be noted that the matrix elements for x and x^2 in the eigenstates of the Hamiltonian [given by (107) and (111) respectively], matches exactly with the corresponding expression given in [25], although the result quoted in [25] is in the eigenstate of the invariant $I(t)$.

The matrix element of y^k in the eigenstates of the Hamiltonian can be obtained similarly, and reads,

$$_H\langle n, m-n|y^k|n, m'-n\rangle_H = \sum_{r=0}^{k}\frac{\pi}{2^k}e^{i\Theta_{0,2r-k}}\,{}^kC_r\delta_{m',m+2r-k}$$
$$\times\ \lambda_n^2(\hbar^{1/2}\rho)^{2n+k+2}\sqrt{m!(m+2r-k)!}$$
$$\times \int_0^\infty dw\; w^{n-m-r+k}e^{-w}L_m^{(n-m)}(w)L_{m+2r-k}^{(n-m-2r+k)}(w). \quad (112)$$

Once again we write down the matrix element of y^k in the eigenstates of the Lewis invariant $I(t)$. This reads

$$\langle n, m-n|y^k|n, m'-n\rangle = \sum_{r=0}^{k}\frac{\pi}{2^k}\,{}^kC_r\delta_{m',m+2r-k}$$
$$\times\ \lambda_n^2(\hbar^{1/2}\rho)^{2n+k+2}\sqrt{m!(m+2r-k)!}$$
$$\times \int_0^\infty dw\; w^{n-m-r+k}e^{-w}L_m^{(n-m)}(w)L_{m+2r-k}^{(n-m-2r+k)}(w). \quad (113)$$

Using (112), we may evaluate the matrix element of y and y^2 in the eigenstate of the Hamiltonian. We find,

$$
\begin{aligned}
{}_H\langle n, m-n|y|n, m'-n\rangle_H &= {}_H\langle n, m-n|y^k\ |_{k=1;\, r=0}\ |n, m'-n\rangle_H \\
&\quad + {}_H\langle n, m-n|y^k\ |_{k=1;\, r=1}\ |n, m'-n\rangle_H \\
&= -\frac{1}{2}(\rho\hbar^{1/2})[\sqrt{m}e^{-i\Theta_{0,1}}\delta_{m',m-1} + \sqrt{m+1}e^{i\Theta_{0,1}}\delta_{m',m+1}]. \qquad (114)
\end{aligned}
$$

$$
\begin{aligned}
{}_H\langle n, m-n|y^2|n, m'-n\rangle_H &= {}_H\langle n, m-n|y^k\ |_{k=2;\, r=0}\ |n, m'-n\rangle_H \\
&\quad + {}_H\langle n, m-n|y^k\ |_{k=2;\, r=1}\ |n, m'-n\rangle_H \\
&\quad + {}_H\langle n, m-n|y^k\ |_{k=2;\, r=2}\ |n, m'-n\rangle_H \\
&= \frac{\hbar\rho^2}{4}\delta_{m',m-2}\sqrt{m(m-1)}e^{-i\Theta_{0,2}} \\
&\quad + \frac{1}{2}\delta_{m,m'}(\hbar\rho^2)(m+n+1) \\
&\quad + \frac{\hbar\rho^2}{4}\delta_{m',m+2}\sqrt{(m+2)(m+1)}e^{i\Theta_{0,2}}. \qquad (115)
\end{aligned}
$$

From the above analysis, we find that even the expression for the matrix element of the operator y^k in the eigenstate of $H(t)$ matches with that found in [25] for $k = 1, 2$, though again they had inappropriately quoted the results in the eigenstate of the Lewis invariant.

5.2 Analysis of the Expectation Value of Energy

As we have already seen from (5), in order to calculate the expectation value of energy one needs the expectation values $\langle {p_1}^2\rangle$, $\langle {p_2}^2\rangle$, $\langle {x_1}^2\rangle$, $\langle {x_2}^2\rangle$, $\langle p_1 x_2\rangle$ and $\langle p_2 x_1\rangle$. As we have seen in the previous subsection, our calculated generalized expressions for matrix elements ${}_H\langle n, m-n|x^k\ |_{k=1;\, r=0}\ |n, m'-n\rangle_H$ and ${}_H\langle n, m-n|y^k\ |_{k=1;\, r=0}\ |n, m'-n\rangle_H$ matched exactly with the calculations in [25] for $k = 1, 2$. Hence, we use the matrix elements quoted in the said work to calculate the following expectation values,

$$
\begin{aligned}
\langle x_j^2\rangle &= \frac{\rho^2}{2}(n+m+1)\ ;\ \langle p_j^2\rangle = \frac{1}{2}\left(\frac{1}{\rho^2} + \frac{\dot{\rho}^2}{a^2}\right)(n+m+1) \\
\langle x_j\, p_k\rangle &= \frac{1}{2}\epsilon_{jk}(m-n)\ ; \qquad (116)
\end{aligned}
$$

where $j, k = 1, 2$ and $\epsilon_{jk} = -\epsilon_{kj}$ with $\epsilon_{12} = 1$. So, the expectation value of energy $\langle E_{n,m-n}(t)\rangle$ with respect to energy eigenstate $\psi_{n,m-n}(r, \theta, t)$ can be expressed as,

$$
\begin{aligned}
\langle E_{n,m-n}(t)\rangle &= \frac{1}{2}(n+m+1)\left[b(t)\rho^2(t) + \frac{a(t)}{\rho^2(t)} + \frac{\dot{\rho}^2(t)}{a(t)}\right] + c(t)(n-m)\ . \\
&= \frac{1}{2}\Bigg[(n+m+1)\left(b(t)\rho^2(t) + \frac{a(t)}{\rho^2(t)} + \frac{\dot{\rho}^2(t)}{a(t)}\right) \\
&\quad + (n-m)\left(\frac{f(t)\Omega(t)}{M} + \frac{M\omega^2(t)\theta(t)}{f(t)}\right)\Bigg]. \qquad (117)
\end{aligned}
$$

It is interesting to note that even when the frequency of oscillation $\omega \to 0$, the expectation value of energy is non-zero. This is because all the three parameters of the Hamiltonian $a(t)$, $b(t)$ and $c(t)$ are finite even as $\omega \to 0$, as is clear from the (6), (7) and (8). Now we will proceed to study the time-dependent behaviour of $\langle E_{n,m-n}(t)\rangle$ for various types of damping.

5.2.1 Exponentially Decaying Solution

For the exponentially decaying solution given by (47), the energy expectation value takes the following form,

$$\langle E_{n,m-n}(t)\rangle = (n+m+1)\mu^2\Delta + c(t)\,(n-m) \tag{118}$$

where we have set the constant ξ^2 to unity and used the constraint relation given by (48).

⟨*A*⟩ Solution Set-Ia For this case we consider $f(t) = 1$ and $\omega(t) = \omega_0\, e^{-\Gamma t/2}$. The expectation value of energy for the ground state has the following expression,

$$\langle E_{n,-n}(t)\rangle \;=\; (n+1)\mu^2\Delta + n\left[\sqrt{\frac{\Delta\, exp\,(\Gamma t) - M{\omega_0}^2\, exp\,(-\Gamma t)}{M}} + \omega_0\, exp\,(-\Gamma t/2)\sqrt{M\sigma\, exp\,(-\Gamma t) - 1}\,\right]. \tag{119}$$

From (119), we see that the expectation value of the energy becomes complex beyond a certain time limit. The condition for getting the expectation value of energy to be real is as follows,

$$M\,\sigma\, e^{-\Gamma t} \geq 1 \;\Rightarrow\; t \leq \frac{ln(M\,\sigma)}{\Gamma} \tag{120}$$

$$e^{2\Gamma t} \geq \frac{M\omega_0^2}{\Delta} \;\Rightarrow\; t \geq \frac{1}{2\Gamma}\ln\left(\frac{M\omega_0^2}{\Delta}\right). \tag{121}$$

This gives the following range of t in which the expectation value of energy is real,

$$\frac{1}{2\Gamma}\ln\left(\frac{M\omega_0^2}{\Delta}\right) \leq t \leq \frac{ln(M\,\sigma)}{\Gamma}\,. \tag{122}$$

It is to be noted that the above restriction on time t is consistent with the hermiticity of the Hamiltonian operator (5). For $t > ln(M\,\sigma)/\,\Gamma$, we observe from (55) that $\theta(t)$ becomes complex and hence the time dependent coefficient $c(t)$ in (8) becomes complex. Similarly, for $t < \frac{1}{2\Gamma}\ln\left(\frac{M\omega_0^2}{\Delta}\right)$, we observe from (56) that $\Omega(t)$ becomes complex and hence the time dependent coefficient $c(t)$ in (8) once again becomes complex. This in turn implies that the Hamiltonian operator (5) ceases to be hermitian. Hence the condition on time for getting real expectation value of energy is compatible with the hermiticity of the Hamiltonian operator since it is hermitian only in this time range.

We see from Fig. 1, that the energy initially decays but then increases with time. This is because for large time at which $exp\,(-\Gamma t/2) \approx 0$, the approximated expression of energy reads

$$E_{n,-n}(t) \approx\; (n+1)\mu^2\Delta + n\sqrt{\frac{\Delta\, exp\,(\Gamma t)}{M}} \tag{123}$$

which is still increasing with time. The reason for the increase of energy with time is the form of the coefficient $b(t)$ in the Hamiltonian. Although the coefficient $a(t)$ is exponentially decaying with time, the coefficient $b(t)$ exponentially increases with time in order to satisfy EP equation. However, since there is an upper limit of time within which the energy remains real, so the energy remains finite within the allowed time interval.

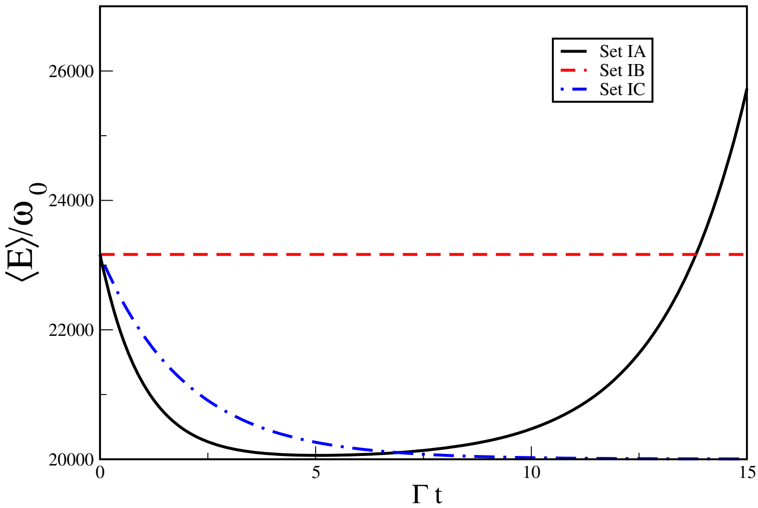

Fig. 1 A study of the variation of expectation value of energy, scaled by $\frac{1}{\omega_0}$ ($\frac{\langle E\rangle}{\omega_0}$) in order to make it dimensionless, as we vary Γt (again a dimensionless quantity). Here we consider mass M=1, μ=1,Δ=10^7, σ=10^7, ω_0=10^3 and Γ=1 in natural units. The expectation value of energy $\langle E\rangle$ is calculated for exponentially decaying Hamiltonian parameters when $\langle A\rangle$ Set-IA $f(t) = 1$ and $\omega(t) = \omega_0 e^{-\Gamma t/2}$; $\langle B\rangle$ Set-IB $f(t) = e^{-\Gamma t}$ and $\omega(t) = \omega_0$ and $\langle C\rangle$ Set-IC $f(t) = e^{-\Gamma t}$ and $\omega(t) = \omega_0 e^{-\Gamma t/2}$. While for $\langle A\rangle$ the energy first decreases, then increases with time, for $\langle B\rangle$ the energy remains constant as we vary time. For $\langle C\rangle$ the energy decays off with time

Here we set $f(t) = e^{-\Gamma\, t}$ and $\omega(t) = \omega_0$. With this the energy expression for the ground state takes the form,

$$\langle E_{n,-n}(t)\rangle = (n+1)\mu^2\Delta + n\left[\sqrt{\frac{\Delta - M{\omega_0}^2}{M}} + \omega_0\sqrt{M\sigma - 1}\right]. \tag{124}$$

We note from Fig. 1, that the expectation value of the energy remarkably remains constant as we vary time, as is observed from (124). This must be because the effect of the exponentially decaying Hamiltonian coefficient $a(t)$ and damping term $f(t)$ gets balanced out by the exponentially increasing Hamiltonian coefficient $b(t)$.

⟨C⟩ Solution Set-Ic Here we set $f(t) = e^{-\Gamma\, t}$ and $\omega(t) = \omega_0\, e^{-\Gamma\, t/2}$. With this the expectation value of the energy expression takes the form,

$$\langle E_{n,-n}(t)\rangle = (n+1)\mu^2\Delta + n\left[\sqrt{\frac{\Delta - M{\omega_0}^2\, exp\,[-\Gamma t]}{M}} + \omega_0\, exp\,(-\Gamma t/2)\sqrt{M\sigma - 1}\right]. \tag{125}$$

The above expression gives a very nice decaying expression for the expectation value of energy with respect to time, and finally approaching a constant value in the limit $t \to \infty$. This behaviour is also exhibited in the nature of the plot of variation of the expectation value of energy with time seen in Fig. 1.

5.2.2 Rationally Decaying Solution

In this case the expectation value of energy for $k = 2$ reads

$$E_{n,-n}(t) = \frac{(n+1)}{2(\Gamma t + \chi)} \left[2\left(\frac{\sigma}{\mu^2} + \Delta\mu^2 \right) + \frac{\mu^2 \Gamma^2}{8\sigma} \right] + n \left[\frac{\omega_0}{\Gamma t + \chi} \sqrt{\frac{4\sigma M}{(\Gamma t + \chi)^2} - 1} + \sqrt{\frac{\Delta}{M} - \frac{\omega_0^2}{(\Gamma t + \chi)^2}} \right]. \tag{126}$$

Note that although it has a nice decaying property like the damping case on commutative plane, there is an upper bound of time above which the energy ceases to be real. The upper bound on time reads,

$$4\sigma M \geq (\Gamma t + \chi)^2 \Rightarrow t \leq \frac{1}{\Gamma}(2\sqrt{M\sigma} - \chi). \tag{127}$$

In fact, the energy is real in the following range of time,

$$\frac{1}{\Gamma}\left(\sqrt{\frac{M}{\Delta}}\omega_0 - \chi \right) \leq t \leq \frac{1}{\Gamma}(2\sqrt{M\sigma} - \chi). \tag{128}$$

The reason for the energy being real in this time range is the same as in the case of the exponentially decaying solution. The Hamiltonian operator (5) is hermitian only in this time range since outside this time range, the NC parameters $\theta(t)$ and $\Omega(t)$ become imaginary and so, the time dependent coefficient $c(t)$ in (8) becomes complex. As a result, the Hamiltonian operator ceases to be hermitian.

From Fig. 2, we see indeed the expectation value of energy $\langle E \rangle$ decays with time following power law as expected for the rationally decaying solutions.

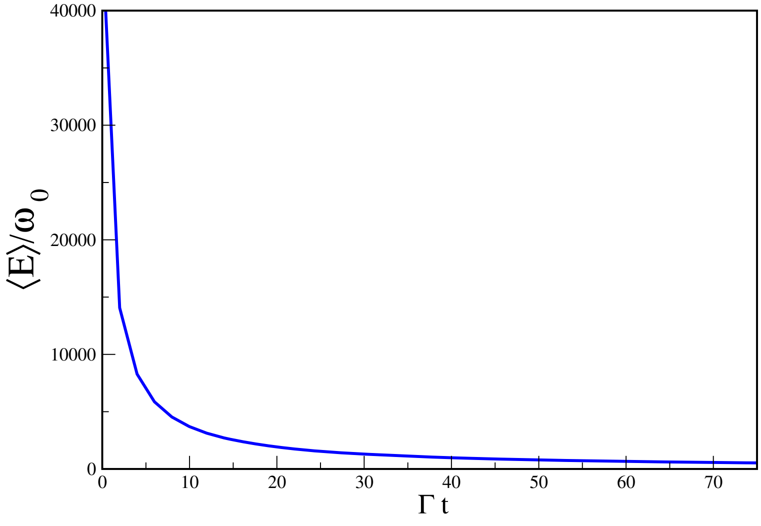

Fig. 2 A study of the variation of expectation value of energy, scaled by $\frac{1}{\omega_0}$ $(\frac{\langle E \rangle}{\omega_0})$ in order to make it dimensionless, as we vary Γt (again a dimensionless quantity). Here we consider mass M=1, μ=1,Δ=10^7, σ=10^7, ω_0=10^3, $\chi = 1$ and Γ=1 in natural units. The expectation value of energy $\langle E \rangle$ is calculated for rationally decaying Hamiltonian parameters. We consider $f(t) = 1$ and $\omega(t) = \frac{\omega_0}{(\Gamma t + \chi)}$

5.2.3 Elementary Solution

For the elementary solution set, the expectation value of the energy reads,

$$\langle E_{n,-n}(t)\rangle = \frac{1}{2}(n+1)\left[\left(\Delta\mu^2 + \frac{\sigma}{\mu^2}\right)\frac{1}{(\Gamma\, t+\chi)^2} + \frac{\mu^2\Gamma^2}{\sigma}\right] + n\left[\frac{\omega_0\sqrt{M\sigma - 1}}{(\Gamma\, t+\chi)} + \frac{1}{(\Gamma\, t+\chi)}\sqrt{\frac{\Delta}{M\,(\Gamma\, t+\chi)^2} - \omega_0^2}\right]. \quad (129)$$

Further, the constraint relation $\Delta\mu^4 = \xi^2\sigma$ results in the following form for the expectation value of energy (setting $\xi^2 = 1$),

$$\langle E_{n,-n}(t)\rangle = \frac{1}{2}(n+1)\left[\frac{2\sigma}{\mu^2(\Gamma\, t+\chi)^2} + \frac{\mu^2\Gamma^2}{\sigma}\right] + n\left[\frac{\omega_0\sqrt{M\sigma - 1}}{(\Gamma\, t+\chi)} + \frac{1}{(\Gamma\, t+\chi)}\sqrt{\frac{\Delta}{M\,(\Gamma\, t+\chi)^2} - \omega_0^2}\right]. \quad (130)$$

This expression also provides an upper bound of the time limit above which the expectation value of energy would become complex. This upper bound reads,

$$\frac{\Delta}{M\,(\Gamma\, t+\chi)^2} \geq \omega_0^2 \Rightarrow t \leq \frac{1}{\Gamma}\left[\frac{1}{\omega_0}\sqrt{\frac{\Delta}{M}} - \chi\right]. \quad (131)$$

Once again the above condition is compatible with the hermiticity of the Hamiltonian operator (5) as it ceases to be hermitian above this time since $\Omega(t)$ becomes complex resulting in the time dependent coefficient $c(t)$ in (8) being complex in this case.

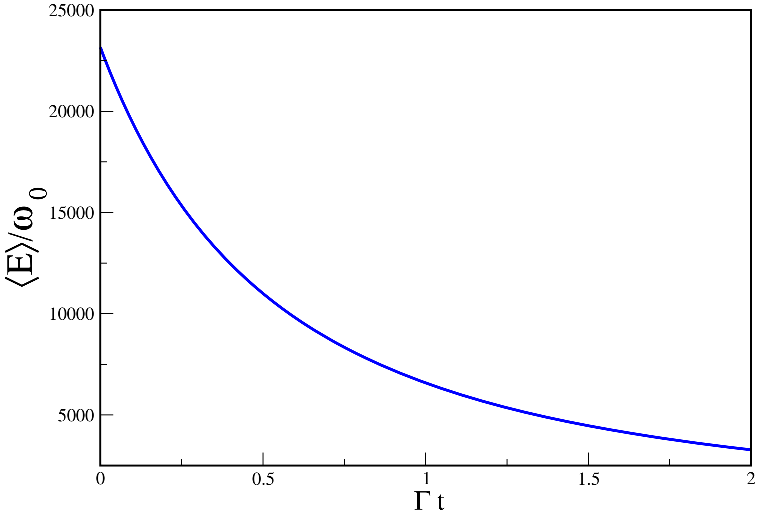

Fig. 3 A study of the variation of expectation value of energy, scaled by $\frac{1}{\omega_0}$ ($\frac{\langle E\rangle}{\omega_0}$) in order to make it dimensionless, as we vary Γt (again a dimensionless quantity). Here we consider mass M=1, μ=1,Δ=10^7, σ=10^7, ω_0=10^3, $\chi = 1$ and Γ=1 in natural units. The expectation value of energy $\langle E\rangle$ is calculated for elementarily decaying Hamiltonian parameters. We consider $f(t) = 1$ and $\omega(t) = \frac{\omega_0}{(\Gamma\, t+\chi)}$

In Fig. 3, we observe that the expectation value of energy again undergoes a power law decay with time for the elementary solution.

6 Conclusion

We now summarize our results. In this paper we have considered a two-dimensional damped harmonic oscillator in noncommutative space with time dependent noncommutative parameters. We map this system in terms of commutative variables by using a shift of variables connecting the noncommutative and commutative space, known in the literature as Bopp-shift. We have then obtained the exact solution of this time dependent system by using the well known Lewis invariant which in turn leads to a non-linear differential equation known as the Ermakov-Pinney equation. We first obtain the Lewis invariant in Cartesian coordinates. We then make a transformation to polar coordinates and write down our results in these coordinates. Doing so, we use the operator approach to obtain the eigenstates of the invariant. With this background in place, we make various choices of the parameters in the problem which in turn leads to solutions for the time dependent noncommutative parameters. We have considered three different sets of choices for which solutions have been obtained, namely, exponentially decaying solutions, rationally decaying solutions and elementary solutions. Interestingly, the solutions obtained make it possible to integrate the phase factor exactly thereby giving an exact solution for the eigenstates of the Hamiltonian. We have then computed the matrix elements of operators raised to a finite integer power in both the eigenstates of the Hamiltonian as well as the Lewis invariant. From these results, we are able to compute the expectation value of the Hamiltonian. Expectedly, the expectation value of the energy varies with time. For the exponentially decaying solutions, we get three kinds of behaviour corresponding to the choices of the damping factor and the frequency of the oscillator. For the case where the damping factor is set to unity and the frequency of the oscillator decays with time, the expectation value of the energy first decreases with time and then increases. The reason for this behaviour is due to the particular form of the solutions of the Ermakov-Pinney equation which fixes the forms of the noncommutative parameters. It is these time dependent forms of the noncommutative parameters that results in the above mentioned behaviour of the expectation value of the energy with time. In this case, we also observe that there is a time range for which the energy expectation value is real. For the case where the damping factor has a decaying part and the frequency of the oscillator is a constant, we observe that the expectation value of the energy remarkably remains constant with time. This must be the case because the effect of the exponentially decaying coefficient in the Hamiltonian and the damping term gets balanced out by the exponentially increasing coefficient in the Hamiltonian. For the case where both the damping term as well as the frequency of the oscillator decays with time, we find an exponentially decaying behaviour of the expectation value of the energy. For the rationally decaying and the elementary solution, we observe a power law decay of the energy expectation value with time together with a time range outside which the energy expectation value ceases to be real. Investigating these cases of damped oscillators, we conclude that the behaviour corresponding to the exponentially decaying solution, where both the frequency and damping term are decaying exponentially with time, is similar to a damped oscillator in commutative space.

Acknowledgements MD would like to thank Ms. Riddhi Chatterjee and Ms.Rituparna Mandal for their helpful assistance to operate the software Mathematica.

Data Availability Data sharing is not applicable to this article as no new data were created or analysed in this study.

References

1. Lewis, H.R., Riesenfeld, W.B. Jr..: J. Math. Phys. **10**, 1458 (1969)
2. Lewis, H.R. Jr..: J. Math. Phys. **9**, 1976 (1968)
3. Lewis, H.R. Jr.: Phys. Rev. Lett. **18**, 510 (1967). Erratum Phys. Rev. Lett. **18**, 636 (1967)
4. Abdalla, M.S., Ismael, N.A.: Int. J. Theor. Phys. **48**, 2757 (2009)
5. Pedrosa, I.A., de Lima, D.: Int. J. Mod. Phys. B **28**, 1450177 (2014)
6. Jannussis, A., Bartzis, B.: Phys. Lett. A **129**, 263 (1988)
7. Pedrosa, I.A.: J. Math. Phys. **28**, 2662 (1987)
8. Pedrosa, I.A.: Rev. Bras. Fis. **19**(3), 502 (1989)
9. Lawson, L.M., Avossevou, G.Y.H., Gouba, L.: J. Math. Phys. **59**, 112101 (2018)
10. Doplicher, S., Fredenhagen, K., Roberts, J.E.: Commun. Math. Phys. **172**, 187 (1995)
11. Amati, D., Ciafaloni, M., Veneziano, G.: Phys. Lett. B **216**, 41 (1989)
12. Seiberg, N., Witten, E.: J. High Energy Phys. **09**, 032 (1999)
13. Rovelli, C.: Living Rev. Relativity **11**, 5 (2008)
14. Bigatti, D., Susskind, L.: Phys. Rev. D **62**, 066004 (2000)
15. Dayi, O.F., Jellal, A.: J. Math. Phys. **43**, 4592 (2002)
16. Chakraborty, B., Gangopadhyay, S., Saha, A.: Phys. Rev. D **70**, 107707 (2004)
17. Scholtz, F.G., Chakraborty, B.: Phys. Rev. D **71**, 085005 (2005)
18. Scholtz, F.G., Chakraborty, B., Gangopadhyay, S., Govaerts, J.: J. Phys. A **38**, 9849 (2005)
19. Chakraborty, B., Gangopadhyay, S., Hazra, A.G., Scholtz, F.G.: J. Phys. A **39**, 9557 (2006)
20. Banerjee, R., Gangopadhyay, S., Modak, S.K.: Phys. Lett. B **686**, 181 (2010)
21. Saha, A., Gangopadhyay, S., Saha, S.: Phys. Rev. D **83**, 025004 (2011)
22. Saha, A., Gangopadhyay, S., Saha, S.: Phys. Rev. D **97**, 044015 (2018)
23. Smailagic, A., Spallucci, E.: J. Phys. A **36**, L467 (2003)
24. Gangopadhyay, S., Scholtz, F.G.: Phys. Rev. Lett. **102**, 241602 (2009)
25. Dey, S., Fring, A.: Phys. Rev. D **90**, 084005 (2014)
26. Streklas, A.: Physica A **385**, 124 (2007)
27. Reuter, M.: Phys. Rev. D **57**, 971 (1998)
28. Reuter, M., Saueressig, F.: Phys. Rev. D **65**, 065016 (2002)
29. Reuter, M., Saueressig, F.: Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety, Cambridge Monographs on Mathematical Physics. https://doi.org/10.1017/9781316227596 (2020)
30. Ermakov, V.: Univ. Izv. Kiev. **20**, 1 (1880)
31. Pinney, E.: Proc. Am. Math. Soc. **1**, 681 (1950)
32. Caldirola, P.: Nuovo Cimento **18**, 393 (1941)
33. Kanai, E.: Prog. Theor. Phys. **3**, 440 (1948)
34. Mezincescu, L.: Star Operation in Quantum Mechanics. arXiv:0007046[hep-th] (2020)
35. Arfken, G.B., Weber, H.J.: Mathematical Methods For Physicists. Academic Press, Inc. (2020)
36. Nikiforov, A.F., Uvarov, V.B.: Special Function of Mathematical Physics. Birkhäuser, Basel, Switzerland (1988)
37. Chiellini, A.: Bolletino dell'Unione Matematica Italiana **10**, 301 (1931)