Genuinely nonlocal product bases: Classification and entanglement-assisted discrimination

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An orthogonal product basis of a composite Hilbert space is genuinely nonlocal if the basis states are locally indistinguishable across every bipartition. From an operational point of view, such a basis corresponds to a separable measurement that cannot be implemented by local operations and classical communication unless all the parties come together in a single location. In this work we classify genuinely nonlocal product bases into different categories. Our classification is based on the state elimination property of the set via orthogonality-preserving measurements when all the parties are spatially separated or different subsets of the parties come together. We then study local state discrimination protocols for several such bases with additional entangled resources shared among the parties. Apart from consuming less entanglement than teleportation-based schemes, our protocols indicate operational significance of the proposed classification and exhibit nontrivial use of genuine entanglement in the local state discrimination problem.

I. INTRODUCTION

The superposition principle lies at the core of quantum mechanics, which leads to several no-go results in quantum information theory such as the no-cloning [1] and the no-deleting theorem [2]. It also gives rise to the concept of nonorthogonal states for which perfect discrimination is never possible. The origin of the quantum state discrimination problem can be traced back to an initial attempt to formalize information processing with optical quantum devices [3–5]. Although sets of mutually orthogonal states can always be perfectly distinguished by some global measurements, the situation may change dramatically for a set of such multipartite quantum states if the spatially separated parties are allowed to perform only local operations assisted with classical communication (LOCC). In a seminal work Bennett *et al.* provided examples of mutually orthogonal product states that are indistinguishable under LOCC given one copy of the state [6]. They coined the term quantum nonlocality without entanglement[1] for this phenomenon as the states allow local preparation (with some preshared strategy) but prohibit perfect local discrimination. Importantly, local indistinguishability turns out to be a crucial primitive for a number of distributed quantum protocols, namely, quantum data hiding [9,10] and quantum secret sharing [11–13].

The result of Bennett *et al.* [6] has motivated overwhelming research interest in generic local state discrimination problems, i.e., the task of optimal discrimination of multiparty states, not necessarily product states, by means of LOCC [14–55]. Very recently, Halder *et al.* introduced a nontrivial generalization of the quantum nonlocality without entanglement phenomenon [56]. They provided examples of three-qutrit and three-ququad product bases that are not distinguishable even if (any) two of the three parties come together, i.e., each of these product bases can be prepared locally, but to distinguish them either all three parties need to come together or entangled resources must be shared across all bipartitions. Quite naturally, one can define such a feature as genuine quantum nonlocality without entanglement. The existence of the product bases of Ref. [56] has important operational consequences. While on the one hand they constitute a nontrivial primitive for multipartite information-theoretic protocols, on the other hand they correspond to multipartite separable measurements that require entanglement resources across all bipartitions for implementing those measurements.

In this work we classify genuinely nonlocal product bases (GNPBs) of multipartite quantum systems. The classification is based on whether such a basis is reducible, i.e., allows elimination of a state or states from the set under orthogonality-preserving measurements (OPMs) when all the parties are spatially separated, or some parties are allowed to come together. We then show that this way of classification has interesting operational consequences. We provide an example of multipartite GNPBs that allow local elimination of some states under OPM even when all the parties are separated and subsequently they require entanglement across one bipartite cut only for perfect discrimination of the states. In other words, local elimination makes it adequate to consume less

[1]Note that this nonlocal feature is different from the concept of quantum nonlocality as established in another seminal work by Bell [7]. A multipartite input-output correlation is called nonlocal if it is not compatible with the classical description of local realism (see [8] for a review on Bell nonlocality). In the quantum world such correlations can only result from multipartite entangled states.

entanglement for perfect discrimination of the states in those GNPBs. We then provide examples of some tripartite GNPB which are locally irreducible when all three parties are in separate locations but reducible if two of the parties are together; however, it requires entanglement across every bipartite cut for perfect discrimination. Such a GNPB is weaker than the GNPB of [56] as the latter does not allow any elimination of states under nontrivial OPM even when any two parties come together. We then provide different entanglement-assisted protocols for perfect discrimination of several GNPBs. The protocols studied are resource efficient as they consume less entanglement in comparison to the teleportation-based protocols. Interestingly, one of our protocols exhibits nontrivial and advantageous use of genuine entanglement in a local state discrimination problem.

The paper is organized as follows. In Sec. II we briefly discuss the notation and several required preliminary concepts. Sections III and IV consist of the main contributions of the present work. In Sec. V we summarize the results and discuss some open problems.

II. PRELIMINARIES AND NOTATION

Throughout the paper we will use standard notation and terminology that are commonly used in quantum information theory. All the systems we consider are finite dimensional and thus the associated Hilbert spaces are isomorphic to some complex Euclidean spaces $\mathbb{C}^d$, with $d \in \mathbb{N}$ denoting the dimension of the system. Composite quantum systems are associated with the tensor product of the corresponding subsystems' Hilbert spaces; an n-partite quantum system is associated with the Hilbert space $\bigotimes_{i=1}^n \mathbb{C}^{d_i}$, where $\mathbb{C}^{d_i}$ corresponds to the ith subsystem. For our purpose we start with recalling the following definition.

Definition 1. Nonlocal product bases. Consider an n-partite quantum system with Hilbert space $\bigotimes_{i=1}^n \mathbb{C}^{d_i}$. An orthogonal product basis (OPB) $\mathbb{B}_{nl} \equiv \{|\psi\rangle_j = \bigotimes_{i=1}^n |\alpha\rangle_j^i | j = 1, \ldots, \Pi_{i=1}^n d_i\}$ is called nonlocal if the states in $\mathbb{B}_{nl}$ cannot be perfectly distinguished by LOCC when all the parties are spatially separated.

Bipartite and multipartite examples of such bases were first constructed in [6] for $(\mathbb{C}^3)^{\otimes 2}$ and $(\mathbb{C}^2)^{\otimes 3}$ Hilbert spaces. However, for the second example the states cannot be discriminated under LOCC when all three parties are spatially separated, but can be done perfectly when two of the parties come together. This observation leads to a stronger notion of nonlocality without entanglement.

Definition 2. Genuinely nonlocal product bases. A multipartite OPB $\mathbb{B}_{gnl} \equiv \{|\psi\rangle_j = \bigotimes_{i=1}^n |\alpha\rangle_j^i | j = 1, \ldots, \Pi_{i=1}^n d_i\} \subset \bigotimes_{i=1}^n \mathbb{C}^{d_i}$ is called genuinely nonlocal if the states in $\mathbb{B}_{gnl}$ cannot be perfectly distinguished by LOCC even if any $n-1$ parties are allowed to come together.

Reference [56] provides examples of such product bases for three-qutrit and three-ququad quantum systems. At this point, let us briefly discuss local protocols. These are generally multiround protocols. For the multipartite case, depending on the scenario, whether all parties are separated or some groups of parties are allowed to come together, in a particular round each party individually and/or some as a group perform local quantum operations and communicate the classical outcomes to other parties and/or other groups. Depending on these communications, other parties (and/or groups) further choose their actions and the protocol goes on. Since there are multiple rounds, it is in general difficult to mathematically characterize the set of LOCC. Some interesting topological behaviors of the LOCC set have been identified in [57]. While discriminating a set of mutually orthogonal multipartite product states by such a LOCC protocol, in a given round either the given state must be identified or some of the states must be eliminated. If the latter happens, then for perfect discrimination the remaining postmeasurement states should be mutually orthogonal so that the subsequent steps can be carried out. This leads to the following definition.

Definition 3. Nontrivial orthogonality-preserving measurement. A measurement performed to distinguish a set of mutually orthogonal quantum states is called an orthogonality-preserving measurement if after the measurement the states remain mutually orthogonal. Furthermore, such a measurement is called nontrivial if all the measurement effects constituting the OPM are not proportional to the identity operator; otherwise it is trivial.

Definition 3 subsequently leads to the concept of a locally irreducible set. i.e., a set of mutually orthogonal multipartite quantum states from which it is not possible to eliminate one or more states by orthogonality-preserving local measurements. Although local irreducibility sufficiently ensures locally indistinguishable, the former is not a requirement for the latter. It turns out that the examples of [56] exhibit a strong nonlocal behavior as not only are those product bases GNPBs, but they are indeed locally irreducible even if any two parties come together.

While additional entangled states are supplied as a resource among the parties along with their operational power LOCC, it may be possible to perfectly distinguish a GNPB. An immediate such protocol follows from quantum teleportation [58]. If the involved parties share sufficient entanglement so that they can teleport their respective subsystems to one of the parties, then they can perfectly discriminate the states by performing suitable measurement. Since entanglement is a costly resource under the operational paradigm of LOCC, any protocol consuming less entanglement is always desirable. The first instance of such protocols for a class of locally indistinguishable product states was proposed by Cohen [59]. For instance, Bennett's two-qutrit nonlocal product basis (NPB) can be perfectly distinguished by LOCC with additional one-ebit entanglement, whereas the teleportation-based protocol requires a two-qutrit maximally entangled state, i.e., log 3 ebit. Here we use the unit ebit, so the logarithm is taken with respect to base 2. Cohen's result motivates further research in identifying efficient use of entanglement in the local state discriminating problem [60–69].

In this work we also study efficient state discrimination protocols for several GNPBs. Note that, in the multipartite scenario, different types of entangled resource may be supplied as there exist different inequivalent types of entanglement. For instance, in the tripartite scenario different pairs of parties can be supplied with a two-qubit maximally entangled state, i.e., Einstein-Podolsky-Rosen (EPR) state, $|\phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in (\mathbb{C}^2)^{\otimes 2}$ or they can be supplied with a three-qubit Greenberger-Horne-Zeilinger (GHZ) state

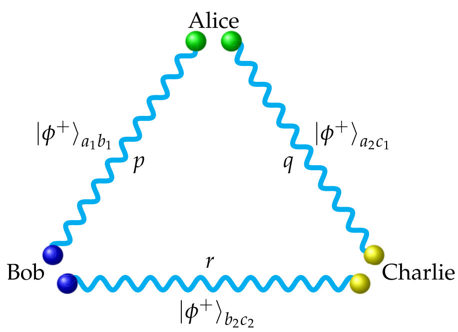

FIG. 1. Spatial configuration of the resource state $\{(p, |\phi^+\rangle_{\mathcal{AB}}); (r, |\phi^+\rangle_{\mathcal{BC}}); (q, |\phi^+\rangle_{\mathcal{CA}})\}$. The indices $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, and $c_1, c_2 \in \mathcal{C}$.

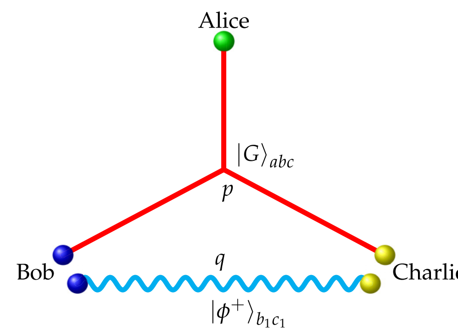

FIG. 2. Spatial configuration of the resource state $\{(p, |G\rangle_{\mathcal{ABC}}); (q, |\phi^+\rangle_{\mathcal{BC}})\}$.

$|G\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \in (\mathbb{C}^2)^{\otimes 3}$. Here we consider two different configurations of entanglement resources.

Configuration 1, $\{(p, |\phi^+\rangle_{\mathcal{AB}}); (r, |\phi^+\rangle_{\mathcal{BC}}); (q, |\phi^+\rangle_{\mathcal{CA}})\}$, with p, q, and r taking non-negative values, means that on average an amount p of the two-qubit maximally entangled state is consumed between Alice and Bob and similarly, in another two pairs, the amounts q and r of the EPR state are consumed while discriminating a GNPB (see Fig. 1). Whenever discriminating a GNPB of $(\mathbb{C}^d)^{\otimes 3}$, a successful discrimination protocol with this resource configuration will be more efficient than the corresponding teleportation-based protocol if $(p + q + r) < 2\log_2 d$.

Configuration 2, $\{(p, |G\rangle_{\mathcal{ABC}}); (q, |\phi^+\rangle_\star)\}$, with p and q taking non-negative values and the star denoting one of the pairs from $\{\mathcal{AB}, \mathcal{BC}, \mathcal{CA}\}$, means that while discriminating a GNPB, an amount p of the three-qubit GHZ state is consumed in addition to an amount q of the EPR state shared between one of the three pairs (see Fig. 2). Note that, to distribute a three-qubit GHZ state among Alice, Bob, and Charlie two copies of the EPR state shared between two pairs, say, between Alice and Bob and between Alice and Charlie, are required; Alice prepares a GHZ state in her laboratory and teleports two subsystems to Bob and Charlie, respectively. However, the process might be irreversible, as there is no known local protocol via which it is possible to get back two copies of two-qubit Bell states from a GHZ state [70].

III. CLASSIFICATION OF GNPBS

The states in an n-partite GNPB cannot be perfectly distinguished under LOCC even if any $n-1$ parties are allowed to come together. However, it may be possible that while some parties come together, they can eliminate some states from the set under local measurement which keeps the post-measurement states orthogonal. Based on how many parties are required to come together for such elimination, we can classify the GNPBs into different types. In the following we will discuss this classification with explicit examples. Though the classification can be generalized for arbitrary number of parties, we will mainly restrict our study to tripartite Hilbert spaces.

A. Type-I GNPB

Such a GNPB is locally reducible even when all the parties are separated, i.e., some subset of states can be eliminated under nontrivial local OPM.

Example. Consider the quantum system with Hilbert space $(\mathbb{C}^4)^{\otimes 3}$ shared among Alice, Bob, and Charlie. Computational bases for $\mathbb{C}^4$ will be denoted by $\{|i\rangle\}_{i=0}^3$. We will use the shorthand notation $|\alpha\rangle_A|\beta\rangle_B|\gamma\rangle_C$ for the state $|\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C$ and will avoid the party index where possible. To construct the required GNPB, first consider the set of states

$$\mathcal{S} \equiv \{|3\rangle_A|\beta\rangle_{BC}, |\beta\rangle_{AB}|3\rangle_C\}, \tag{1}$$

where the $|\beta\rangle$ are the states belonging in the two-qutrit NPB of Ref. [6], i.e., $|\beta\rangle \in \mathcal{B} \equiv \{|0\rangle|\eta_\pm\rangle, |\eta_\pm\rangle|2\rangle, |2\rangle|\xi_\pm\rangle, |\xi_\pm\rangle|0\rangle, |1\rangle|1\rangle\}$, with $|\eta_\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|\xi_\pm\rangle := (|1\rangle \pm |2\rangle)/\sqrt{2}$. The definitions of $|\eta_\pm\rangle$ and $|\xi_\pm\rangle$ are kept the same throughout the paper. Also consider the following product states:

$$\begin{aligned}\mathcal{R} \equiv \{&|3\rangle|0\rangle|3\rangle, |3\rangle|1\rangle|3\rangle, |3\rangle|2\rangle|3\rangle, |3\rangle|3\rangle|0\rangle, |3\rangle|3\rangle|1\rangle, |3\rangle|3\rangle|2\rangle, |3\rangle|3\rangle|3\rangle, |2\rangle|0\rangle|0\rangle,\\ &|2\rangle|0\rangle|1\rangle, |2\rangle|0\rangle|2\rangle, |2\rangle|1\rangle|0\rangle, |2\rangle|1\rangle|1\rangle, |2\rangle|1\rangle|2\rangle, |2\rangle|2\rangle|0\rangle, |2\rangle|2\rangle|1\rangle, |2\rangle|2\rangle|2\rangle,\\ &|2\rangle|3\rangle|0\rangle, |2\rangle|3\rangle|1\rangle, |2\rangle|3\rangle|2\rangle, |2\rangle|3\rangle|3\rangle, |0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|1\rangle, |0\rangle|0\rangle|2\rangle, |0\rangle|1\rangle|0\rangle,\\ &|0\rangle|1\rangle|1\rangle, |0\rangle|1\rangle|2\rangle, |0\rangle|2\rangle|0\rangle, |0\rangle|2\rangle|1\rangle, |0\rangle|2\rangle|2\rangle, |0\rangle|3\rangle|0\rangle, |0\rangle|3\rangle|1\rangle, |0\rangle|3\rangle|2\rangle,\\ &|0\rangle|3\rangle|3\rangle, |1\rangle|0\rangle|0\rangle, |1\rangle|0\rangle|1\rangle, |1\rangle|0\rangle|2\rangle, |1\rangle|1\rangle|0\rangle, |1\rangle|1\rangle|1\rangle, |1\rangle|1\rangle|2\rangle, |1\rangle|2\rangle|0\rangle,\\ &|1\rangle|2\rangle|1\rangle, |1\rangle|2\rangle|2\rangle, |1\rangle|3\rangle|0\rangle, |1\rangle|3\rangle|1\rangle, |1\rangle|3\rangle|2\rangle, |1\rangle|3\rangle|3\rangle\}.\end{aligned} \tag{2}$$

These 46 states in $\mathcal{R}$ along with the states in $\mathcal{S}$ form an OPB in $(\mathbb{C}^4)^{\otimes 3}$. Since the set $\mathcal{B}$ is a NPB in $(\mathbb{C}^3)^{\otimes 2}$, the set of tripartite states $\{|3\rangle_A|\beta\rangle_{BC} \mid |\beta\rangle \in \mathcal{B}\}$ are locally indistinguishable across the $B|AC$ cut as well as the $C|AB$ cut. Similarly, the set of states $\{|\beta\rangle_{AB}|3\rangle_C \mid |\beta\rangle \in \mathcal{B}\}$ is locally indistinguishable across the $A|BC$ and $B|AC$ cuts. As the considered OPB contains both these subsets of states, we have the following proposition.

Proposition 1. The set of orthogonal states $\mathbb{B}_I(4,3) \equiv \mathcal{S} \cup \mathcal{R}$ is a GNPB in $(\mathbb{C}^4)^{\otimes 3}$. However, the set is locally reducible even when all the parties are separated.

Construction of such GNPBs is straightforward for an arbitrary number of parties. At this point it is important to note that the local indistinguishability arises due to the *twisted* states $|\beta\rangle \in \mathcal{B}$, which are obtained from linear superposition of computational states. In other words, the quantum superposition principle plays a key role in the manifestation of quantum nonlocality without entanglement phenomenon. While discriminating the states in $\mathbb{B}_I(4,3)$, any of the parties can perform nontrivial OPM to eliminate certain states even when all of them are separated. For instance, Alice performs a local measurement $M \equiv \{|3\rangle\langle 3|, \mathbb{I}_4 - |3\rangle\langle 3|\}$. If the measurement result corresponds to the projector $|3\rangle\langle 3|$, then the given state must be one of the following states:

$$|3\rangle \Rightarrow \{|3\rangle|\beta\rangle, |3\rangle|0\rangle|3\rangle, |3\rangle|1\rangle|3\rangle, |3\rangle|2\rangle|3\rangle, |3\rangle|3\rangle|0\rangle, |3\rangle|3\rangle|1\rangle, |3\rangle|3\rangle|2\rangle, |3\rangle|3\rangle|3\rangle\}. \quad (3)$$

Otherwise it is one of the remaining states. Since the measurement considered is an OPM, after this step the entanglement-assisted discrimination protocol can be carried out. The outcome provides nontrivial information about which cut is needed to share bipartite entanglement. If the outcome corresponds to the projector $|3\rangle_A\langle 3|$, then entanglement is required to share between Bob and Charlie; otherwise it should be shared between Alice and Bob. In other words, local elimination makes it possible to consume entanglement across one bipartite cut only for perfect discrimination of the states.

B. Type-II GNPB

Such a GNPB is locally irreducible when all the parties are separated, i.e., no local elimination is possible preserving orthogonality among the postmeasurement states.

Example. From the GNPB $\mathbb{B}_I(4,3)$ every party can locally eliminate some states by performing a OPM that discriminates the subspace spanned by $|3\rangle$ vs subspace spanned by $\{|0\rangle, |1\rangle, |2\rangle\}$. One will obtain a GNPB of type II if this local elimination can be stopped. For this purpose, take the states $\{|0\rangle|3\rangle|2\rangle, |0\rangle|3\rangle|3\rangle\} \subset \mathbb{B}_I(4,3)$. Consider now a new OPB that contains the locally twisted product states $\{|0\rangle|3\rangle|\chi_\pm\rangle\}$ instead of the states $\{|0\rangle|3\rangle|2\rangle, |0\rangle|3\rangle|3\rangle\}$, where $|\chi_\pm\rangle := (|2\rangle \pm |3\rangle)/\sqrt{2}$. As a consequence, Charlie is no longer able to discriminate between the subspaces spanned by $|3\rangle$ and $\{|0\rangle, |1\rangle, |2\rangle\}$ and thus he cannot eliminate any state via OPM. Similarly, the twisted product states $\{|2\rangle|\chi_\pm\rangle|2\rangle\}$ stop Bob and $\{|\chi_\pm\rangle|3\rangle|1\rangle\}$ stop Alice from eliminating any state via nontrivial OPM. Thus we have the following proposition.

Proposition 2. The set of states

$$\mathbb{B}_{II}(4,3) := \{\mathbb{B}_I(4,3) \setminus \{|0\rangle|3\rangle|2\rangle, |0\rangle|3\rangle|3\rangle, |2\rangle|2\rangle|2\rangle, |2\rangle|3\rangle|2\rangle, |2\rangle|3\rangle|1\rangle, |3\rangle|3\rangle|1\rangle\} \cup \{|0\rangle|3\rangle|\chi_\pm\rangle, |2\rangle|\chi_\pm\rangle|2\rangle, |\chi_\pm\rangle|3\rangle|1\rangle\}\}$$

is a GNPB in $(\mathbb{C}^4)^{\otimes 3}$. Furthermore, the set is locally irreducible when all the parties are specially separated.

Like the type-I basis, in this case also it is also possible to generalize the construction for an arbitrary number of parties. Here we find that the set of multipartite nonlocal product states constructed in Ref. [48] also possesses a feature similar to our type-II basis. Clearly, the GNPB $\mathbb{B}_{\mathrm{II}}(4,3)$ requires entanglement across every bipartition for perfect discrimination. In other words, if two of the parties come together, then also nonlocality persists. We make two important observations. (i) The local twist plays an important role in the construction of a new class of GNPBs: It can increase the entanglement cost of distinguishing a product basis by LOCC. (ii) The examples $\mathbb{B}_{\mathrm{I}}(4,3)$ and $\mathbb{B}_{\mathrm{II}}(4,3)$ also exhibit the operational implication of local elimination via nontrivial OPM as perfect discrimination of the first set requires an entangled resource only in one cut and the latter demands entanglement in more than one cut.

So far we have constructed GNPBs for Hilbert spaces with local subsystem dimension 4. Naturally, a question arises regarding such constructions in lower-dimensional cases. Recall that all product bases in $\mathbb{C}^2 \otimes \mathbb{C}^d$ are locally distinguishable [20]. Therefore, for a GNPB to exist the minimum dimension is $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. However, the technique used above and the technique used in [48] are not applicable to construct GNPBs in the minimum dimension. In the following we provide an example of the GNPB in $(\mathbb{C}^3)^{\otimes 3}$.

Proposition 3. The set of states

$$\mathbb{B}_{II}(3,3) \equiv \{|0\rangle|\eta_\pm\rangle|\xi_\pm\rangle, |\eta_\pm\rangle|2\rangle|\xi_\pm\rangle, |2\rangle|\xi_\pm\rangle|\eta_\pm\rangle, |\eta_\pm\rangle|\xi_\pm\rangle|0\rangle, |\xi_\pm\rangle|0\rangle|\eta_\pm\rangle, |\xi_\pm\rangle|\eta_\pm\rangle|2\rangle, |k\rangle|k\rangle|k\rangle \mid k \in \{0,1,2\}\}$$

is a GNPB of type II in $(\mathbb{C}^3)^{\otimes 3}$.

When the three parties are spatially separated, it is not possible to eliminate any state by OPM from the above set and hence the set is locally indistinguishable. This can be easily proved by the technique described in [56,68]. We are yet to prove local indistinguishability of the above set across every bipartition. For that, first note that set $\mathcal{B}$ is present between any two pairs in the above construction. For instance, consider the subset of states

$$\{|\xi_\pm\rangle|0\rangle|\eta_\pm\rangle, |\xi_\pm\rangle|\eta_\pm\rangle|2\rangle, |\eta_\pm\rangle|2\rangle|\xi_\pm\rangle, |\eta_\pm\rangle|\xi_\pm\rangle|0\rangle, |1\rangle|1\rangle|1\rangle\}.$$

The presence of $\mathcal{B}$ between Bob and Charlie is evident here. Furthermore, Alice's states $|\eta_\pm\rangle$, $|\xi_\pm\rangle$, and $|1\rangle$ are tagged. These tagged states are not all mutually orthogonal due to the presence of a local twist. Moreover, the twisted states saturate the local dimension of Alice, i.e., $|\eta_\pm\rangle$ covers the subspace spanned by $|0\rangle$ and $|1\rangle$ whereas $|\xi_\pm\rangle$ covers the subspace spanned by $|1\rangle$ and $|2\rangle$. As a result, even if Alice comes together with either Bob or Charlie, it is not possible to perfectly

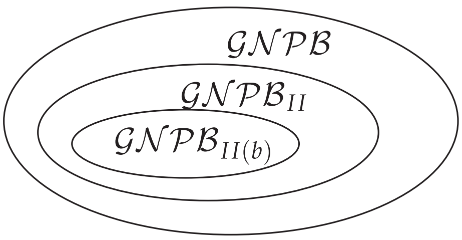

FIG. 3. The set $\mathcal{GNPB}_{\text{II(b)}}$ of all type-II(b) GNPBs is a proper subset of the set $\mathcal{GNPB}_{\text{II}}$ of all type-II GNPBs which is again a proper subset of the set $\mathcal{GNPB}$ of all GNPBs, i.e., $\mathcal{GNPB}_{\text{II(b)}} \subset \mathcal{GNPB}_{\text{II}} \subset \mathcal{GNPB}$.

distinguish this set of states. Similar argument holds in the case of other bipartitions and consequently $\mathbb{B}_{\text{II}}(3,3)$ turns out to be a GNPB of type II. Note that this particular basis appears in a recent work for a different purpose [71]. There the aim was to construct a genuinely entangled subspace such that all density matrices supported on it are genuinely entangled. The entanglement-assisted discrimination protocol of this basis is described in the next section.

As already discussed, from a type-II GNPB, no state can be eliminated under OPM when all the parties are separated. However, when two parties come together the power of state elimination under OPM may increase. This leads us to the further classification of the type-II GNPBs.

Type-II(a) GNPB. Such a GNPB is not locally reducible when all the parties are in separate location; however, when two of the parties come together it is possible to eliminate some states through nontrivial OPM. A careful observation reveals that the GNPB in Proposition 2 is an example of such a GNPB (evidently this follows from Remark 2 in Appendix D).

Type-II(b) GNPB. Such a GNPB is locally irreducible even if any two parties come together. Examples of such GNPBs are constructed in [56] for Hilbert spaces $(\mathbb{C}^3)^{\otimes 3}$ and $(\mathbb{C}^4)^{\otimes 3}$. Here we redraft the three-qutrit example.

Proposition 4. From [56] we have the set of states

$$\begin{aligned}\mathbb{B}_{\text{II(b)}}(3,3) \equiv \{&|0\rangle|1\rangle|\eta_\pm\rangle, |1\rangle|\eta_\pm\rangle|0\rangle, |\eta_\pm\rangle|0\rangle|1\rangle,\\ &|0\rangle|2\rangle|\kappa_\pm\rangle, |2\rangle|\kappa_\pm\rangle|0\rangle, |\kappa_\pm\rangle|0\rangle|2\rangle,\\ &|1\rangle|2\rangle|\eta_\pm\rangle, |2\rangle|\eta_\pm\rangle|1\rangle, |\eta_\pm\rangle|1\rangle|2\rangle,\\ &|2\rangle|1\rangle|\kappa_\pm\rangle, |1\rangle|\kappa_\pm\rangle|2\rangle, |\kappa_\pm\rangle|2\rangle|1\rangle,\\ &|k\rangle|k\rangle|k\rangle \mid k \in \{0,1,2\}\}\end{aligned}$$

in a GNPB of type II(b) in $(\mathbb{C}^3)^{\otimes 3}$, with $|\kappa_\pm\rangle := (|0\rangle \pm |2\rangle)/\sqrt{2}$.

Clearly type II(b) is the strongest form of GNPB from the perspective of local elimination. The above classification thus introduces a hierarchical relation as depicted in Fig. 3.

IV. ENTANGLEMENT-ASSISTED DISCRIMINATION

In this section we study entanglement-assisted discrimination protocols for the GNPBs discussed earlier. First we consider the three-qutrit GNPBs. Note that, in $(\mathbb{C}^3)^{\otimes 3}$, two pairs of two-qutrit maximally entangled states, i.e., $2\log 3$ ebits, distributed between Alice and Bob and between Alice and Charlie always lead to perfect discrimination for any genuinely nonlocal basis. Therefore, any protocol that consumes fewer than $2\log 3$ ebits is nontrivial and resource efficient. The following proposition constitutes such a nontrivial protocol.

Proposition 5. The entanglement resource $\{(1, |\phi^+(3)\rangle_{\mathcal{AB}}); (0, |\phi^+\rangle_{\mathcal{BC}}); (1, |\phi^+\rangle_{\mathcal{CA}})\}$ is sufficient for local discrimination of the GNPBs $\mathbb{B}_{\text{II}}(3,3)$ and $\mathbb{B}_{\text{II(b)}}(3,3)$, where $|\phi^+(3)\rangle := (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3} \in \mathbb{C}^3 \otimes \mathbb{C}^3$.

Using the two-qutrit maximally entangled state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$, Bob first teleports his subsystem to Alice. Entanglement consumed at this step amounts to $\log 3$ ebits. After that one-ebit entanglement shared between Alice and Charlie suffices for perfect discrimination of these GNPBs (see Appendix A for the detailed protocol). Therefore, in total, $(\log 3 + 1)$-ebit entanglement is consumed in this protocol, which is strictly less than the amount consumed in the protocol using teleportation in both arms. However, in this protocol the teleportation scheme is used in one arm. We now show that even more efficient protocols are possible to discriminate these GNPBs.

Proposition 6. The entanglement resource $\{(1, |\phi^+\rangle_{\mathcal{AB}}); (0, |\phi^+\rangle_{\mathcal{BC}}); (1, |\phi^+\rangle_{\mathcal{CA}})\}$ sufficiently discriminates the GNPB $\mathbb{B}_{\text{II}}(3,3)$ when all the parties are separated.

See Appendix B for the protocol. Clearly the entanglement consumed in this protocol is strictly less than $1 + \log 3$ ebits. The resource state used in the protocol, i.e., the state $|\phi^+\rangle_{AB} \otimes |\phi^+\rangle_{\mathcal{CA}}$, exists in the Hilbert space $\mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Naturally, the question arises whether a lower-dimensional resource from the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ will suffice for perfect discrimination. At this point we observe that a protocol similar to Proposition 6 that uses a three-qubit GHZ state $|G\rangle$ or a three-qubit W state $|W\rangle := (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ does not lead to perfect discrimination of the basis $\mathbb{B}_{\text{II}}(3,3)$.

Here we want to point out some important observations. For discriminating the NPB of $(\mathbb{C}^3)^{\otimes 2}$, Cohen pointed out that in his protocol two-qubit maximally entangled states are the necessary resource. If instead of the two-qubit maximally entangled resource a partially entangled state $\lambda_0|00\rangle + \lambda_1|11\rangle$, with $\lambda_0 \neq \lambda_1$, is provided as the resource, then at some stage the protocol leads to nonorthogonal states and consequently the protocol does not succeed perfectly. Furthermore, he also gave an impression that for any successful protocol a two-qubit maximally entangled state may be the necessary resource; however, this assertion is yet to be proven. If it indeed turns out to be the case, then the three-qubit W state cannot be a sufficient resource for perfect discrimination of the basis $\mathbb{B}_{\text{II}}(3,3)$. On the other hand, if no local protocol exists that simultaneously generates two EPR pairs shared between Alice and Bob and between Alice and Charlie, respectively, from a three-qubit GHZ state, then it also cannot be the sufficient resource for perfect discrimination of $\mathbb{B}_{\text{II}}(3,3)$. Therefore, at this point the question remains whether the resource mentioned in Proposition 6 is indeed the necessary resource for perfect discrimination of the set $\mathbb{B}_{\text{II}}(3,3)$.

Using the same resource as of Proposition 6, we then proceed to discriminate the set $\mathbb{B}_{\text{II(b)}}(3,3)$ following an analogous protocol. We find that with this amount of resource, though some state can be eliminated, after a certain stage the protocol cannot be further extended up to perfect discrimination. However, if an additional entanglement resource is provided, then a perfect discrimination protocol is possible as stated in the following proposition (a detailed protocol is provided in Appendix C).

Proposition 7. The entanglement resource $\{(1, |\phi^+\rangle_{\mathcal{AB}}); (\frac{8}{27}, |\phi^+\rangle_\star); (1, |\phi^+\rangle_{\mathcal{CA}})\}$ is sufficient for perfect local discrimination of the GNPB $\mathbb{B}_{\mathrm{II(b)}}(3,3)$, where the star is one of the pairs from $\{\mathcal{AB}, \mathcal{BC}, \mathcal{CA}\}$.

The average entanglement used in this protocol is therefore $1 + 1 + \frac{8}{27} \cong 2.296$ ebits, which is less than $1 + \log 3 \cong 2.585$ ebits consumed in Proposition 5. Note that the resource state exists either in the Hilbert space $\mathbb{C}^8 \otimes \mathbb{C}^4 \otimes \mathbb{C}^2$ (if the additional entanglement in Proposition 7 is shared between Alice and Bob) or in the Hilbert space $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ (if the additional entanglement is shared between Bob and Charlie). At this point the question remains open whether a lower-dimensional resource state from $\mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ will result in a successful discrimination protocol for $\mathbb{B}_{\mathrm{II(b)}}(3,3)$ or the resource in Proposition 7 is necessary.

Let us now consider entanglement-assisted discrimination protocol for the GNPBs in $(\mathbb{C}^4)^{\otimes 3}$. As already discussed, for the GNPB $\mathbb{B}_{\mathrm{I}}(4,3)$ in Proposition 1, the entanglement resource only in one cut is sufficient for perfect discrimination. Of course, in which cut the entangled resource needs to be shared is determined after the first elimination step under OPM. Furthermore, in this case, since the genuine indistinguishability arises due to the presence of the $(\mathbb{C}^3)^{\otimes 2}$ NPB between Alice and Bob (i.e., the set of states $\{|\beta\rangle_{AB}|3\rangle_C\}$) and between Bob and Charlie (i.e., the set of states$\{|3\rangle_A|\beta\rangle_{BC}\}$), Cohen's protocol [59] ensures that a two-qubit maximally entangled state shared between *AB* or *BC* (decided accordingly after the first elimination step) is sufficient for perfect discrimination even though the local dimension for each party is 4.

Consider now the GNPB $\mathbb{B}_{\mathrm{II}}(4,3)$ in Proposition 2. Since this one is a GNPB of type II, no state can be eliminated under OPM while all the parties are spatially separated and consequently an entangled resource across every bipartition is required in this case. Interestingly, here we find that if the three parties share a three-qubit GHZ state then they can start the discrimination protocol. However, as stated in the following proposition, the perfect discrimination protocol we obtain requires an additional EPR pair along with the GHZ resource.

Proposition 8. The entanglement resource $\{(1, |G\rangle_{\mathcal{ABC}}); (\frac{1}{8}, |\phi^+\rangle_\star)\}$ is sufficient for perfect local discrimination of the GNPB $\mathbb{B}_{\mathrm{II}}(4,3)$ in Proposition 2, where the star is one of the pairs from $\{\mathcal{AB}, \mathcal{BC}, \mathcal{CA}\}$.

See Appendix D for a detailed protocol. The above protocol exhibits nontrivial use of multipartite entanglement in the local state discrimination protocol. Moreover, we observe that instead of the GHZ state, if Alice and Bob start the protocol by sharing an EPR state, then for perfect discrimination an additional $\frac{11}{16}$ of an ebit is required to be shared between Bob and Charlie (see the Remark 2 in Appendix D). This indicates an advantage of a genuine entangled resource over its bipartite counterpart in the state discrimination problem. However, conclusive proof of this assertion requires establishing the necessary requirement of entangled resources in different such protocols, which we leave here as an open question for future research.

V. SUMMARY AND OPEN PROBLEMS

The phenomenon of strong quantum nonlocality without entanglement introduced in Ref. [56] motivated us to look for other techniques to construct genuinely nonlocal product bases. As a result, in this work we have classified GNPBs into different categories. Via this classification we have addressed an important question regarding the requirement of a multipartite entangled resource state for perfect discrimination of a GNPB. Interestingly, we have found that elimination of certain states from the original set by performing orthogonality-preserving measurements may help to reduce the entanglement consumption for perfect discrimination. We have also presented entanglement-assisted local discrimination protocols of several GNPBs. These protocols are resource efficient as they consume less entanglement than a teleportation-based protocol. We have also addressed an open problem raised in Ref. [56]. The authors there left open the possibility of the existence of a cheaper resource than that of a teleportation-based scheme for perfect discrimination of a strong nonlocal basis. One of our protocols provides an affirmative answer to this question. Moreover, we have discrimination protocols for GNPBs with different types and configurations of entangled resources. Interestingly, we found a strong indication of a genuine entanglement advantage over bipartite entanglement for discrimination of some GNPBs.

Our study also raises a few important questions. First of all, the question of optimality of the entangled resources used in our discrimination protocols remains open. Clearly, this study should shed light on the optimal resource requirement for the implementation of the separable measurement corresponding to these GNPBs. Furthermore, it is intriguing to study whether the classification of GNPBs induces a hierarchy among the corresponding separable measurements.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF PROPOSITION 5

1. Discrimination of $\mathbb{B}_{\mathrm{II}}(3,3)$

The GNPB of Proposition 3 is given by

$$\mathbb{B}_{\mathrm{II}}(3,3) \equiv \{|\psi(\pm,\pm)\rangle_1 := |0\rangle_A|\eta_\pm\rangle_B|\xi_\pm\rangle_C,$$
$$|\psi(\pm,\pm)\rangle_2 := |\eta_\pm\rangle_A|2\rangle_B|\xi_\pm\rangle_C,$$
$$|\psi(\pm,\pm)\rangle_3 := |2\rangle_A|\xi_\pm\rangle_B|\eta_\pm\rangle_C,$$
$$|\psi(\pm,\pm)\rangle_4 := |\eta_\pm\rangle_A|\xi_\pm\rangle_B|0\rangle_C,$$
$$|\psi(\pm,\pm)\rangle_5 := |\xi_\pm\rangle_A|0\rangle_B|\eta_\pm\rangle_C,$$
$$|\psi(\pm,\pm)\rangle_6 := |\xi_\pm\rangle_A|\eta_\pm\rangle_B|2\rangle_C,$$

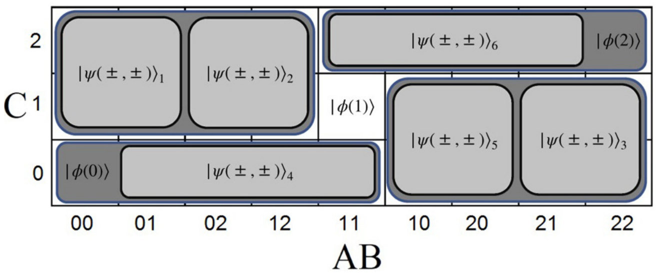

FIG. 4. Tile structure of the GNPB $\mathbb{B}_{\text{II}}(3,3)$ in $AB|C$ cut. This particular tile structure is similar to that of the $\mathbb{C}^3 \otimes \mathbb{C}^3$ tile UPB.

$$|\phi(k)\rangle := |k\rangle_A|k\rangle_B|k\rangle_C \mid k \in \{0,1,2\}\}. \tag{A1}$$

Using the entanglement resource $|\phi^+(3)\rangle$, Bob teleports his subsystem to Alice. Thus, without loss of generality, after this step we can think that they are in the same laboratory and we will use the subindex $\tilde{A}$ for this joint part. While Alice and Bob are together the states in $\mathbb{B}_{\text{II}}(3,3)$ have the tile structure in Fig. 4.

To discriminate the state Charlie shares $|\phi^+\rangle$ with $\tilde{A}$. Therefore, the initial state is

$$|\psi\rangle_{\tilde{A}C} \otimes |\phi^+\rangle_{ac}, \tag{A2}$$

where $|\psi\rangle_{\tilde{A}C}$ is one of the states from $\mathbb{B}_{\text{II}}(3,3)$ and this allows representation as in Fig. 5 (see [59] for details of this representation).

For shorthand notation we will use $|ij\rangle$ to denote $|\mathbf{3i}+\mathbf{j}\rangle$. Now the discrimination protocol proceeds as follows.

Step 1. Charlie performs the measurement

$$\mathcal{N} \equiv \{N := \mathbb{P}[(|0\rangle,|1\rangle)_C;|0\rangle_c]+\mathbb{P}[|2\rangle_C;|1\rangle_c], \overline{N} := \mathbb{I}-N\},$$

where $\mathbb{P}[(|i\rangle,|j\rangle)_\$;(|k\rangle,|l\rangle)_\#] := (|i\rangle\langle i|+|j\rangle\langle j|)_\$ \otimes (|k\rangle\langle k|+|l\rangle\langle l|)_\#$; this definition is applicable for all the protocols. Suppose the outcome corresponds to N clicks.

Step 2. Alice performs the measurement

$$\mathcal{K} \equiv \{K_1 := \mathbb{P}[(|\mathbf{3}\rangle,|\mathbf{6}\rangle,|\mathbf{7}\rangle,|\mathbf{8}\rangle)_{\tilde{A}};|0\rangle_a],$$
$$K_2 := \mathbb{P}[(|\mathbf{3}\rangle,|\mathbf{4}\rangle,|\mathbf{6}\rangle,|\mathbf{7}\rangle,|\mathbf{8}\rangle)_{\tilde{A}};|1\rangle_a], K_3 := \mathbb{I}-K_1-K_2\}. \tag{A3}$$

If K_1 clicks, the given state is one of $\{|\psi(\pm,\pm)\rangle_3, |\psi(\pm,\pm)\rangle_5\}$ and this set of states is perfectly LOCC distinguishable. If K_1 clicks, the state is one of $\{|\psi(\pm,\pm)\rangle_6, |\phi(2)\rangle\}$ and again a LOCC distinguishable set; otherwise it is one of the remaining 14 states.

Step 3. Charlie performs the measurement $\mathcal{N}' \equiv \{N' := \mathbb{P}[|0\rangle_C;\mathbb{I}_c], \overline{N}' := \mathbb{I}-N'\}$. If N' clicks, the state is one of $\{|\psi(\pm,\pm)\rangle_4, |\phi(0)\rangle\}$ (a LOCC distinguishable set); otherwise it is one of the nine remaining states.

Step 4. Alice performs the measurement $\mathcal{K}' \equiv \{K' := \mathbb{P}[|\mathbf{4}\rangle_{\tilde{A}};\mathbb{I}_a], \overline{K}' := \mathbb{I}-K'\}$. If K' clicks, the state is $|\phi(1)\rangle$; otherwise it is one of $\{|\psi(\pm,\pm)\rangle_1, |\psi(\pm,\pm)\rangle_2\}$, a LOCC distinguishable set. If in step 1 $\overline{N}$ clicks, then also a similar protocol follows.

2. Discrimination of $\mathbb{B}_{\text{II(b)}}(3,3)$

The GNPB of Proposition 3 is given by

$$\begin{aligned}
\mathbb{B}_{\text{II}(b)}(3,3) \equiv \{&|\alpha(\pm)\rangle_1 := |0\rangle_A|1\rangle_B|\eta_\pm\rangle_C,\\
&|\alpha(\pm)\rangle_2 := |0\rangle_A|2\rangle_B|\kappa_\pm\rangle_C,\\
&|\alpha(\pm)\rangle_3 := |1\rangle_A|2\rangle_B|\eta_\pm\rangle_C,\\
&|\alpha(\pm)\rangle_4 := |2\rangle_A|1\rangle_B|\kappa_\pm\rangle_C,\\
&|\beta(\pm)\rangle_1 := |1\rangle_A|\eta_\pm\rangle_B|0\rangle_C,\\
&|\beta(\pm)\rangle_2 := |2\rangle_A|\kappa_\pm\rangle_B|0\rangle_C,\\
&|\beta(\pm)\rangle_3 := |2\rangle_A|\eta_\pm\rangle_B|1\rangle_C,\\
&|\beta(\pm)\rangle_4 := |1\rangle_A|\kappa_\pm\rangle_B|2\rangle_C,\\
&|\gamma(\pm)\rangle_1 := |\eta_\pm\rangle_A|0\rangle_B|1\rangle_C,\\
&|\gamma(\pm)\rangle_2 := |\kappa_\pm\rangle_A|0\rangle_B|2\rangle_C,\\
&|\gamma(\pm)\rangle_3 := |\eta_\pm\rangle_A|1\rangle_B|2\rangle_C,\\
&|\gamma(\pm)\rangle_4 := |\kappa_\pm\rangle_A|2\rangle_B|1\rangle_C,\\
&|\phi(k)\rangle := |k\rangle_A|k\rangle_B|k\rangle_C \mid k \in \{0,1,2\}\}.
\end{aligned} \tag{A4}$$

While Alice and Bob are together (after Bob teleports his part to Alice using log 3 ebits) the states in $\mathbb{B}_{\text{II(b)}}(3,3)$ have the tile structure in Fig. 6.

Step 1. Charlie performs the measurement

$$\mathcal{N} \equiv \{N := \mathbb{P}[(|0\rangle,|1\rangle)_C;|0\rangle_c]+\mathbb{P}[|2\rangle_C;|1\rangle_c], \overline{N} := \mathbb{I}-N\}.$$

Suppose N clicks.

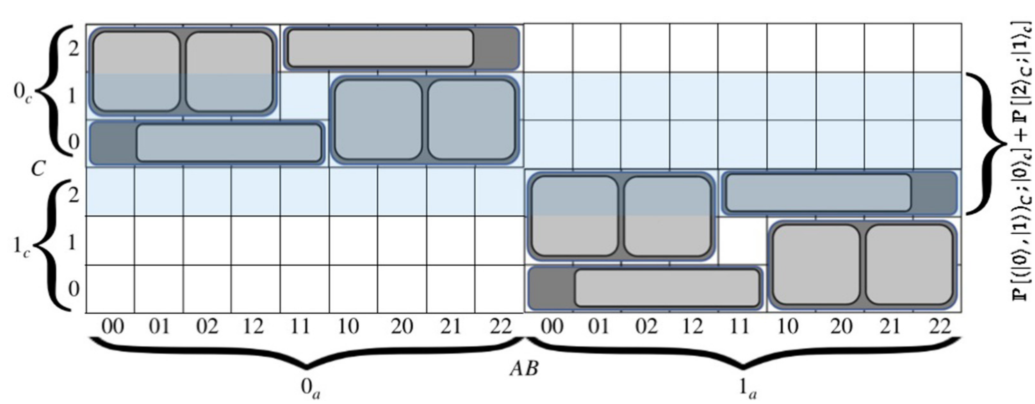

FIG. 5. While Alice and Bob are together, the state $|\psi\rangle_{\tilde{A}C} \otimes |\phi^+\rangle_{ac}$ exists in $\mathbb{C}^{18} \otimes \mathbb{C}^6$. The curly bracket on the right-hand side denotes the measurement effect $N := \mathbb{P}[(|0\rangle,|1\rangle)_C;|0\rangle_c] + \mathbb{P}[|2\rangle_C;|1\rangle_c]$ in step 1.

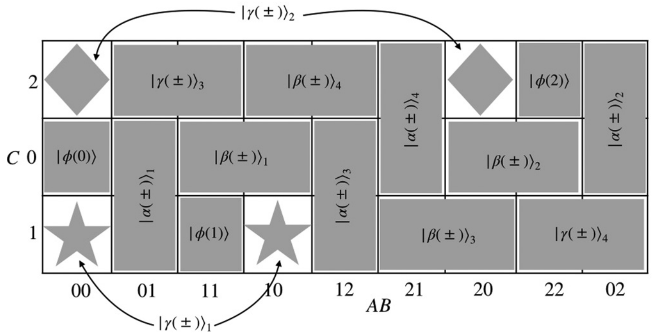

FIG. 6. Tile structure of the GNPB $\mathbb{B}_{\text{II(b)}}(3,3)$ in the $AB|C$ cut. The star- and diamond-shaped tiles contain the states $|\gamma(\pm)\rangle_1$ and $|\gamma(\pm)\rangle_2$), respectively.

Step 2. Alice's measurement and the states corresponding to different outcomes are shown.

$$\begin{aligned}\mathcal{K} \equiv \{&K_1 := \mathbb{P}[(|\mathbf{0}\rangle, |\mathbf{3}\rangle, |\mathbf{4}\rangle)_{\tilde{A}}; |0\rangle_a] \Rightarrow \{|\beta(\pm)\rangle_1, |\gamma(\pm)\rangle_1, |\phi(0)\rangle, |\phi(1)\rangle\}\\ &K_2 := \mathbb{P}[|\mathbf{1}\rangle_{\tilde{A}}; |0\rangle_a] \Rightarrow \{|\alpha(\pm)\rangle_1\},\\ &K_3 := \mathbb{P}[|\mathbf{5}\rangle_{\tilde{A}}; |0\rangle_a] \Rightarrow \{|\alpha(\pm)\rangle_3\},\\ &K_4 := \mathbb{P}[(|\mathbf{0}\rangle, |\mathbf{6}\rangle)_{\tilde{A}}; |1\rangle_a] \Rightarrow \{|\gamma(\pm)\rangle_2\},\\ &K_5 := \mathbb{P}[(|\mathbf{1}\rangle, |\mathbf{4}\rangle)_{\tilde{A}}; |1\rangle_a] \Rightarrow \{|\gamma(\pm)\rangle_3\},\\ &K_6 := \mathbb{P}[(|\mathbf{3}\rangle, |\mathbf{5}\rangle)_{\tilde{A}}; |1\rangle_a] \Rightarrow \{|\beta(\pm)\rangle_4\},\\ &K_7 := \mathbb{P}[|\mathbf{8}\rangle_{\tilde{A}}; |1\rangle_a] \Rightarrow \{|\phi(2)\rangle\},\\ &K_8 := \mathbb{I} - \sum_{i=1}^{7} K_i \Rightarrow \{|\alpha(\pm)\rangle_2, |\alpha(\pm)\rangle_4, |\beta(\pm)\rangle_2, |\beta(\pm)\rangle_3, |\gamma(\pm)\rangle_4\}\}.\end{aligned} \tag{A5}$$

Step 3. If K_1 clicks, Charlie performs the measurement $\mathcal{N}' \equiv \{N' := \mathbb{P}[|0\rangle_C; \mathbb{I}_c], \overline{N}' := \mathbb{I} - N'\}$. If N' clicks, then Alice performs the measurement $\mathcal{K}' \equiv \{K' := \mathbb{P}[|\mathbf{0}\rangle_{\tilde{A}}; \mathbb{I}_a], \overline{K}' := \mathbb{I} - K'\}$; otherwise she performs $\mathcal{K}' \equiv \{K' := \mathbb{P}[|\mathbf{4}\rangle_{\tilde{A}}; \mathbb{I}_a], \overline{K}' := \mathbb{I} - K'\}$. The states corresponding to the outcomes are

$$\begin{aligned}\{N', K' \Rightarrow |\phi(0)\rangle; N', \overline{K}' \Rightarrow \{|\beta(\pm)\rangle_1\}; \overline{N}',\\ K' \Rightarrow |\phi(1)\rangle; \overline{N}', \overline{K}' \Rightarrow \{|\gamma(\pm)\rangle_1\}.\end{aligned} \tag{A6}$$

If K_8 clicks, Charlie performs the measurement $\mathcal{N}' \equiv \{N' := \mathbb{P}[|1\rangle_C; \mathbb{I}_c], \overline{N}' := \mathbb{I} - N'\}$. If N' clicks, then Alice performs the measurement $\mathcal{K}' \equiv \{K' := \mathbb{P}[(|\mathbf{6}\rangle, |\mathbf{7}\rangle)_{\tilde{A}}; \mathbb{I}_a], \overline{K}' := \mathbb{I} - K'\}$; otherwise she performs $\mathcal{K}' \equiv \{K_1' := \mathbb{P}[|\mathbf{7}\rangle_{\tilde{A}}; \mathbb{I}_a], K_2' := \mathbb{P}[|\mathbf{2}\rangle_{\tilde{A}}; \mathbb{I}_a], K_3' := \mathbb{I} - K_1' - K_2'\}$. The states corresponding to the outcomes are

$$\begin{aligned}\{N', K' \Rightarrow \{|\beta(\pm)\rangle_3\}; N', \overline{K}' \Rightarrow \{|\gamma(\pm)\rangle_4\}; \overline{N}',\\ K_1' \Rightarrow \{|\alpha(\pm)\rangle_4\};\\ \overline{N}', K_2' \Rightarrow \{|\alpha(\pm)\rangle_2\}; \overline{N}', K_3' \Rightarrow \{|\beta(\pm)\rangle_2\}.\end{aligned} \tag{A7}$$

In step 1, if $\overline{N}$ clicks instead of N, then also a similar protocol follows.

APPENDIX B: PROOF OF PROPOSITION 6

We need to discriminate the basis $\mathbb{B}_{\text{II}}(3,3)$. Let the EPR state between Alice and Bob be denoted by $|\phi^+\rangle_{a_1b_1}$ and that shared between Alice and Charlie by $|\phi^+\rangle_{a_2c_1}$. Therefore, the initial shared states among them is

$$|\psi\rangle_{ABC} \otimes |\phi^+\rangle_{a_1b_1} \otimes |\phi^+\rangle_{a_2c_1}, \tag{B1}$$

where $|\psi\rangle_{ABC}$ is one of the states from the set $\mathbb{B}_{\text{II}}(3,3)$.

Step 1. Bob performs a measurement

$$\begin{aligned}\mathcal{M} \equiv \{&M := \mathbb{P}[(|0\rangle, |1\rangle)_B; |0\rangle_{b_1}] + \mathbb{P}[|2\rangle_B; |1\rangle_{b_1}],\\ &\overline{M} := \mathbb{I} - M\}\end{aligned}$$

and Charlie performs measurement

$$\mathcal{N} \equiv \{N := \mathbb{P}[(|1\rangle, |2\rangle)_C; |0\rangle_{c_1}] + \mathbb{P}[|0\rangle_C; |1\rangle_{c_1}], \overline{N} := \mathbb{I} - N\}.$$

Suppose the outcomes corresponding to M and N click. The resulting postmeasurement state is therefore

$$\begin{aligned}\{&|\psi(\pm,\pm)\rangle_1 \to |0\rangle_A|\eta_\pm\rangle_B|\xi_\pm\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1},\\ &|\psi(\pm,\pm)\rangle_2 \to |\eta_\pm\rangle_A|2\rangle_B|\xi_\pm\rangle_C|11\rangle_{a_1b_1}|00\rangle_{a_2c_1},\\ &|\psi(\pm,\pm)\rangle_3 \to |2\rangle_A(|1\rangle_B|00\rangle_{a_1b_1}\\ &\qquad \pm |2\rangle_B|11\rangle_{a_1b_1})(|0\rangle_C|11\rangle_{a_2c_1} \pm |1\rangle_C|00\rangle_{a_2c_1}),\\ &|\psi(\pm,\pm)\rangle_4 \to |\eta_\pm\rangle_A(|1\rangle_B|00\rangle_{a_1b_1}\\ &\qquad \pm |2\rangle_B|11\rangle_{a_1b_1})|0\rangle_C|11\rangle_{a_2c_1},\\ &|\psi(\pm,\pm)\rangle_5 \to |\xi_\pm\rangle_A|0\rangle_B|00\rangle_{a_1b_1}(|0\rangle_C|11\rangle_{a_2c_1}\\ &\qquad \pm |1\rangle_C|00\rangle_{a_2c_1}),\\ &|\psi(\pm,\pm)\rangle_6 \to |\xi_\pm\rangle_A|\eta_\pm\rangle_B|2\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1},\\ &|\phi(0)\rangle \to |0\rangle_A|0\rangle_B|0\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1},\\ &|\phi(1)\rangle \to |1\rangle_A|1\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1},\\ &|\phi(2)\rangle \to |2\rangle_A|2\rangle_B|2\rangle_C|11\rangle_{a_1b_1}|00\rangle_{a_2c_1}\}.\end{aligned} \tag{B2}$$

Step 2. Alice performs the measurement

$$\begin{aligned}\mathcal{K} \equiv \{&K_1 := \mathbb{P}[(|0\rangle, |1\rangle)_A; |1\rangle_{a_1}; |0\rangle_{a_2}],\\ &K_2 := \mathbb{P}[|0\rangle_A; |0\rangle_{a_1}; |0\rangle_{a_2}], K_3 := \mathbb{I} - K_1 - K_2\}.\end{aligned}$$

If K_1 clicks, the given state is from the set $\{|\psi(\pm,\pm)\rangle_2\}$, which is LOCC distinguishable. If, K_2 clicks, the state is one of $\{|\psi(\pm,\pm)\rangle_1\}$ (a LOCC distinguishable set); otherwise the state is one of the 19 remaining states.

Step 3. Charlie performs the measurement $\mathcal{N}' \equiv \{N' := \mathbb{P}[|2\rangle_C; \mathbb{I}_{c_1}], \overline{N}' := \mathbb{I} - N'\}$. If N' clicks, the state is one of $\{|\psi(\pm,\pm)\rangle_6, |\phi(2)\rangle\}$, which is perfectly LOCC distinguishable.

Step 4. Bob performs the measurement $\mathcal{M}' \equiv \{M' := \mathbb{P}[|0\rangle_B; \mathbb{I}_{b_1}], \overline{M}' := \mathbb{I} - M'\}$. If M' clicks, the state is one of $\{|\psi(\pm,\pm)\rangle_5, |\phi(0)\rangle\}$, which is again perfectly LOCC distinguishable.

Step 5. Alice performs the measurement $\mathcal{K}' \equiv \{K' := \mathbb{P}[|2\rangle_A; \mathbb{I}_{a_1}; \mathbb{I}_{a_2}], \overline{K}' := \mathbb{I} - K'\}$. If K' clicks, the state is one of $\{|\psi(\pm,\pm)\rangle_3\}$ (a LOCC distinguishable set); otherwise it is from the remaining set of LOCC distinguishable states $\{|\psi(\pm,\pm)\rangle_4, |\phi(1)\rangle\}$.

After step 1, only one case (corresponding to the outcomes M and N) is discussed. For all other cases a similar protocol follows.

APPENDIX C: PROOF OF PROPOSITION 7

We need to discriminate the set $\mathbb{B}_{\text{II(b)}}(3,3)$. The EPR state shared between Alice and Bob is denoted by $|\phi^+\rangle_{a_1b_1}$, that between Alice and Charlie by $|\phi^+\rangle_{a_2c_1}$, and that between Bob and Charlie by $|\phi^+\rangle_{b_2c_2}$. Therefore, the initial shared states among them are

$$|\psi\rangle_{ABC}\otimes|\phi^+\rangle_{a_1b_1}\otimes|\phi^+\rangle_{a_2c_1}\otimes|\phi^+\rangle_{b_2c_2}, \tag{C1}$$

where $|\psi\rangle_{ABC}$ is one of the states from the set $\mathbb{B}_{\text{II(b)}}(3,3)$ which they want to identify by LOCC.

Step 1. Bob performs a measurement

$$\mathcal{M}\equiv\{M:=\mathbb{P}[(|0\rangle,|1\rangle)_B;|0\rangle_{b_1}]+\mathbb{P}[|2\rangle_B;|1\rangle_{b_1}],$$
$$\overline{M}:=\mathbb{I}-M\}.$$

Charlie performs a measurement

$$\mathcal{N}\equiv\{N:=\mathbb{P}[(|0\rangle,|1\rangle)_C;|0\rangle_{c_1}]+\mathbb{P}[|2\rangle_C;|1\rangle_{c_1}],$$
$$\overline{N}:=\mathbb{I}-N\}.$$

Suppose that the outcomes corresponding to M and N click. The resulting postmeasurement state is therefore

$$\begin{aligned}
\{|\alpha(\pm)\rangle_1&\to|0\rangle_A|1\rangle_B|\eta_\pm\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\alpha(\pm)\rangle_2&\to|0\rangle_A|2\rangle_B|11\rangle_{a_1b_1}(|0\rangle_C|00\rangle_{a_2c_1}\\
&\qquad\pm|2\rangle_C|11\rangle_{a_2c_1})|\phi^+\rangle_{b_2c_2},\\
|\alpha(\pm)\rangle_3&\to|1\rangle_A|2\rangle_B|\eta_\pm\rangle_C|11\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\alpha(\pm)\rangle_4&\to|2\rangle_A|1\rangle_B|00\rangle_{a_1b_1}(|0\rangle_C|00\rangle_{a_2c_1}\\
&\qquad\pm|2\rangle_C|11\rangle_{a_2c_1})|\phi^+\rangle_{b_2c_2},\\
|\beta(\pm)\rangle_1&\to|1\rangle_A|\eta_\pm\rangle_B|0\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\beta(\pm)\rangle_2&\to|2\rangle_A(|0\rangle_B|00\rangle_{a_1b_1}\\
&\qquad\pm|2\rangle_B|11\rangle_{a_1b_1})|0\rangle_C|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\beta(\pm)\rangle_3&\to|2\rangle_A|\eta_\pm\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\beta(\pm)\rangle_4&\to|1\rangle_A(|0\rangle_B|00\rangle_{a_1b_1}\\
&\qquad\pm|2\rangle_B|11\rangle_{a_1b_1})|2\rangle_C|11\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\gamma(\pm)\rangle_1&\to|\eta_\pm\rangle_A|0\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\gamma(\pm)\rangle_2&\to|\kappa_\pm\rangle_A|0\rangle_B|2\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\gamma(\pm)\rangle_3&\to|\eta_\pm\rangle_A|1\rangle_B|2\rangle_C|00\rangle_{a_1b_1}|11\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\gamma(\pm)\rangle_4&\to|\kappa_\pm\rangle_A|2\rangle_B|1\rangle_C|11\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\phi(0)\rangle&\to|0\rangle_A|0\rangle_B|0\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\phi(1)\rangle&\to|1\rangle_A|1\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\phi(2)\rangle&\to|2\rangle_A|2\rangle_B|2\rangle_C|11\rangle_{a_1b_1}|11\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2}\}.
\end{aligned}\tag{C2}$$

Step 2. Alice performs the measurement

$$\begin{aligned}
\mathcal{K}\equiv\{&K_1:=\mathbb{P}[|1\rangle_A;|1\rangle_{a_1};|0\rangle_{a_2}],\ K_2:=\mathbb{P}[|2\rangle_A;|1\rangle_{a_1};|1\rangle_{a_2}],\\
&K_3:=\mathbb{P}[(|0\rangle_A,|1\rangle_A);|0\rangle_{a_1};|0\rangle_{a_2}],\ K_4:=\mathbb{I}-K_1-K_2-K_3\}.
\end{aligned}\tag{C3}$$

If K_1 clicks, the given state $|\psi\rangle_{ABC}$ is one of $\{|\alpha(\pm)\rangle_3\}$, which are LOCC discriminable; if K_2 clicks, the state is $|\phi(2)\rangle$; if K_3 clicks, the state is anyone from the following set

$$\begin{aligned}
\{|\alpha(\pm)\rangle_1&\to|0\rangle_A|1\rangle_B|\eta_\pm\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\beta(\pm)\rangle_1&\to|1\rangle_A|\eta_\pm\rangle_B|0\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\gamma(\pm)\rangle_1&\to|\eta_\pm\rangle_A|0\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\phi(0)\rangle&\to|0\rangle_A|0\rangle_B|0\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2},\\
|\phi(1)\rangle&\to|1\rangle_A|1\rangle_B|1\rangle_C|00\rangle_{a_1b_1}|00\rangle_{a_2c_1}|\phi^+\rangle_{b_2c_2}\}.
\end{aligned}\tag{C4}$$

Since all states of the ancillary systems a_1b_1 and a_2c_1 are identical in all cases, they provide no further advantage in discrimination and therefore are redundant. Detaching these ancillary systems along with $|\phi^+\rangle_{b_2c_2}$, the remaining states are the complete product bases corresponding to the shift unextendible product basis (UPB) of $\mathbb{C}^2\otimes\mathbb{C}^2\otimes C^2$ [14]. These states cannot be further discriminated under LOCC. However, the additional resource state $|\phi^+\rangle_{b_2c_2}$ makes it possible to discriminate the above states perfectly.

If K_4 clicks, the given state is one of the $27-(2+1+8)=16$ remaining states. For these states the discriminating protocol goes as follows.

Step 3. Charlie performs $\mathcal{N}'\equiv\{N':=\mathbb{P}[|1\rangle_C;\mathbb{I}_{c_1}],\overline{N}':=\mathbb{I}-N'\}$. If N' clicks, the state is given from $\{|\beta(\pm)\rangle_3,|\gamma(\pm)\rangle_4\}$; otherwise it is one of the 12 remaining states.

Step 4. Bob performs $\mathcal{M}'\equiv\{M':=\mathbb{P}[|1\rangle_B;\mathbb{I}_{b_1}],\overline{M}':=\mathbb{I}-M'\}$. If M' clicks, the state is given from $\{|\alpha(\pm)\rangle_4,|\gamma(\pm)\rangle_3\}$; otherwise it is one of the eight remaining states.

Step 5. Alice performs the measurement

$$\begin{aligned}
\mathcal{K}'\equiv\{&K_1':=\mathbb{P}[|0\rangle_A;|1\rangle_{a_1};\mathbb{I}_{a_2}],\ K_2':=\mathbb{P}[|2\rangle_A;\mathbb{I}_{a_1};|0\rangle_{a_2}],\\
&K_3':=\mathbb{P}[|1\rangle_A;\mathbb{I}_{a_1};\mathbb{I}_{a_2}],\ K_4':=\mathbb{P}[(|0\rangle_A,|2\rangle_A);|0\rangle_{a_1};|1\rangle_{a_2}]\}.
\end{aligned}\tag{C5}$$

If K_1' clicks, the state is one of $\{|\alpha(\pm)\rangle_2\}$; if K_2' clicks, the state is one of $\{|\beta(\pm)\rangle_2\}$; if K_3' clicks, the state is one of $\{|\beta(\pm)\rangle_4\}$; otherwise the state is one of $\{|\gamma(\pm)\rangle_2\}$.

Since the given state is chosen randomly from the set $\mathbb{B}_{\text{II(b)}}(3,3)$, the average entanglement consumption in the above protocol is $1+1+\frac{8}{27}$ ebits. Recall here that after step 1, only one outcome is discussed. Other outcomes are also equally likely and hence the entanglement consumption is actually the average.

APPENDIX D: PROOF OF PROPOSITION 8

The set of states needs to be discriminated is given by

$$\begin{aligned}
\mathbb{B}_{\text{II}}(4,3):=\{&|3\rangle|\beta\rangle,|\beta\rangle|3\rangle,|0\rangle|3\rangle|\chi_\pm\rangle,|2\rangle|\chi_\pm\rangle|2\rangle,|\chi_\pm\rangle|3\rangle|1\rangle,|3\rangle|0\rangle|3\rangle,|3\rangle|1\rangle|3\rangle,|3\rangle|2\rangle|3\rangle,\\
&|3\rangle|3\rangle|0\rangle,|3\rangle|3\rangle|2\rangle,|3\rangle|3\rangle|3\rangle,|2\rangle|0\rangle|0\rangle,|2\rangle|0\rangle|1\rangle,|2\rangle|0\rangle|2\rangle,|2\rangle|1\rangle|0\rangle,|2\rangle|1\rangle|1\rangle,
\end{aligned}$$

$$|2\rangle|1\rangle|2\rangle,\ |2\rangle|2\rangle|0\rangle,\ |2\rangle|2\rangle|1\rangle,\ |2\rangle|3\rangle|0\rangle,\ |2\rangle|3\rangle|3\rangle,\ |0\rangle|0\rangle|0\rangle,\ |0\rangle|0\rangle|1\rangle,\ |0\rangle|0\rangle|2\rangle,$$
$$|0\rangle|1\rangle|0\rangle,\ |0\rangle|1\rangle|1\rangle,\ |0\rangle|1\rangle|2\rangle,\ |0\rangle|2\rangle|0\rangle,\ |0\rangle|2\rangle|1\rangle,\ |0\rangle|2\rangle|2\rangle,\ |0\rangle|3\rangle|0\rangle,$$
$$|0\rangle|3\rangle|1\rangle,\ |1\rangle|0\rangle|0\rangle,\ |1\rangle|0\rangle|1\rangle,\ |1\rangle|0\rangle|2\rangle,\ |1\rangle|1\rangle|0\rangle,\ |1\rangle|1\rangle|1\rangle,\ |1\rangle|1\rangle|2\rangle,$$
$$|1\rangle|2\rangle|0\rangle,\ |1\rangle|2\rangle|1\rangle,\ |1\rangle|2\rangle|2\rangle,\ |1\rangle|3\rangle|0\rangle,\ |1\rangle|3\rangle|1\rangle,\ |1\rangle|3\rangle|2\rangle,\ |1\rangle|3\rangle|3\rangle\}, \tag{D1}$$

where $|\beta\rangle \in \mathcal{B} \equiv \{|0\rangle|\eta_\pm\rangle, |\eta_\pm\rangle|2\rangle, |2\rangle|\xi_\pm\rangle, |\xi_\pm\rangle|0\rangle, |1\rangle|1\rangle\}$. Suppose they share the resource state $(|000\rangle_{abc} + |111\rangle_{abc})/\sqrt{2}$ among them.

Step 1. Bob performs the measurement

$$\mathcal{M} \equiv \{M := \mathbb{P}[(|0\rangle, |1\rangle)_B; |0\rangle_b] + \mathbb{P}[(|2\rangle, |3\rangle)_B; |1\rangle_b], \overline{M} := \mathbb{I} - M\}.$$

Suppose M clicks. The sets of states tagged only with $|000\rangle_{abc}$, only with $|111\rangle_{abc}$, or in an entangled form of these tags are, respectively,

$$|000\rangle_{abc} \Rightarrow \{|3\rangle|0\rangle|\eta_\pm\rangle,\ |3\rangle|\eta_\pm\rangle|2\rangle,\ |3\rangle|1\rangle|1\rangle,\ |0\rangle|\eta_\pm\rangle|3\rangle,\ |\xi_\pm\rangle|0\rangle|3\rangle,\ |1\rangle|1\rangle|3\rangle,$$
$$|3\rangle|0\rangle|3\rangle,\ |3\rangle|1\rangle|3\rangle,\ |2\rangle|0\rangle|0\rangle,\ |2\rangle|0\rangle|1\rangle,\ |2\rangle|0\rangle|2\rangle,\ |2\rangle|1\rangle|0\rangle,\ |2\rangle|1\rangle|1\rangle,$$
$$|2\rangle|1\rangle|2\rangle,\ |0\rangle|0\rangle|0\rangle,\ |0\rangle|0\rangle|1\rangle,\ |0\rangle|0\rangle|2\rangle,\ |0\rangle|1\rangle|0\rangle,\ |0\rangle|1\rangle|1\rangle,\ |0\rangle|1\rangle|2\rangle,$$
$$|1\rangle|0\rangle|0\rangle,\ |1\rangle|0\rangle|1\rangle,\ |1\rangle|0\rangle|2\rangle,\ |1\rangle|1\rangle|0\rangle,\ |1\rangle|1\rangle|1\rangle,\ |1\rangle|1\rangle|2\rangle\}, \tag{D2}$$

$$|111\rangle_{abc} \Rightarrow \{|3\rangle|2\rangle|\xi_\pm\rangle,\ |\eta_\pm\rangle|2\rangle|3\rangle,\ |0\rangle|3\rangle|\chi_\pm\rangle,\ |2\rangle|\chi_\pm\rangle|2\rangle,\ |\chi_\pm\rangle|3\rangle|1\rangle,$$
$$|3\rangle|2\rangle|3\rangle,\ |3\rangle|3\rangle|0\rangle,\ |3\rangle|3\rangle|2\rangle,\ |3\rangle|3\rangle|3\rangle,\ |2\rangle|2\rangle|0\rangle,\ |2\rangle|2\rangle|1\rangle,\ |2\rangle|3\rangle|0\rangle,$$
$$|2\rangle|3\rangle|3\rangle,\ |0\rangle|2\rangle|0\rangle,\ |0\rangle|2\rangle|1\rangle,\ |0\rangle|2\rangle|2\rangle,\ |0\rangle|3\rangle|0\rangle,\ |0\rangle|3\rangle|1\rangle,\ |1\rangle|2\rangle|0\rangle,$$
$$|1\rangle|2\rangle|1\rangle,\ |1\rangle|2\rangle|2\rangle,\ |1\rangle|3\rangle|0\rangle,\ |1\rangle|3\rangle|1\rangle,\ |1\rangle|3\rangle|2\rangle,\ |1\rangle|3\rangle|3\rangle\}, \tag{D3}$$

$$\text{entangled} \Rightarrow \{|3\rangle_A(|1\rangle_B|000\rangle_{abc} \pm |2\rangle_B|111\rangle_{abc})|0\rangle_C,\ |2\rangle_A(|1\rangle_B|000\rangle_{abc} \pm |2\rangle_B|111\rangle_{abc})|3\rangle_C\}. \tag{D4}$$

Step 2. Alice performs the measurement

$$\mathcal{K} \equiv \{K_1 := \mathbb{P}[|0\rangle_A; |0\rangle_a], K_2 := \mathbb{P}[(|0\rangle, |1\rangle)_A; |1\rangle_a], K_3 := \mathbb{I} - K_1 - K_2\}.$$

States corresponding to the outcomes K_1 and K_2 are

$$K_1 \Rightarrow \{|0\rangle|\eta_\pm\rangle|3\rangle,\ |0\rangle|0\rangle|0\rangle,\ |0\rangle|0\rangle|1\rangle,\ |0\rangle|0\rangle|2\rangle,\ |0\rangle|1\rangle|0\rangle,\ |0\rangle|1\rangle|1\rangle,\ |0\rangle|1\rangle|2\rangle\}, \tag{D5}$$

$$K_2 \Rightarrow \{|\eta_\pm\rangle|2\rangle|3\rangle,\ |0\rangle|3\rangle|\chi_\pm\rangle,\ |0\rangle|2\rangle|0\rangle,\ |0\rangle|2\rangle|1\rangle,\ |0\rangle|2\rangle|2\rangle,\ |0\rangle|3\rangle|0\rangle,\ |0\rangle|3\rangle|1\rangle,$$
$$|1\rangle|2\rangle|0\rangle,\ |1\rangle|2\rangle|1\rangle,\ |1\rangle|2\rangle|2\rangle,\ |1\rangle|3\rangle|0\rangle,\ |1\rangle|3\rangle|1\rangle,\ |1\rangle|3\rangle|2\rangle,\ |1\rangle|3\rangle|3\rangle\}. \tag{D6}$$

Both of these sets are LOCC distinguishable. If K_3 clicks, the state is one of the 40 remaining states.

Step 3. Charlie performs the measurement

$$\mathcal{N} \equiv \{N_1 := \mathbb{P}[|2\rangle_C; |0\rangle_c], N_2 := \mathbb{P}[(|1\rangle, |2\rangle)_C; |1\rangle_c], N_3 := \mathbb{I} - N_1 - N_2\}.$$

States corresponding to the outcomes N_1 and N_2 are

$$\{N_1 \Rightarrow \{|3\rangle|\eta_\pm\rangle|2\rangle,\ |2\rangle|0\rangle|2\rangle,\ |2\rangle|1\rangle|2\rangle,\ |1\rangle|0\rangle|2\rangle,\ |1\rangle|1\rangle|2\rangle\},$$
$$N_2 \Rightarrow \{|3\rangle|2\rangle|\xi_\pm\rangle,\ |2\rangle|\chi_\pm\rangle|2\rangle,\ |\chi_\pm\rangle|3\rangle|1\rangle,\ |3\rangle|3\rangle|2\rangle,\ |2\rangle|2\rangle|1\rangle\}\}. \tag{D7}$$

The states corresponding to outcome N_1 are LOCC distinguishable. The LOCC distinguishability of the set of states corresponding to N_2 is discussed later (see Remark 1). If N_3 clicks, the given state is one of the 26 remaining states.

Step 4. Bob performs the measurement

$$\mathcal{M}' \equiv \{M_1' := \mathbb{P}[|0\rangle_B; \mathbb{I}_b], M_2' := \mathbb{P}[|3\rangle_B; \mathbb{I}_b], M_3' := \mathbb{I} - M_1' - M_2'\}.$$

States corresponding to the outcomes M_1' and M_2' are

$$\{M_1' \Rightarrow \{|3\rangle|0\rangle|\eta_\pm\rangle,\ |\xi_\pm\rangle|0\rangle|3\rangle,\ |3\rangle|0\rangle|3\rangle,\ |2\rangle|0\rangle|0\rangle,\ |2\rangle|0\rangle|1\rangle,\ |1\rangle|0\rangle|0\rangle,\ |1\rangle|0\rangle|1\rangle\},$$
$$M_2' \Rightarrow \{|3\rangle|3\rangle|0\rangle,\ |3\rangle|3\rangle|3\rangle,\ |2\rangle|3\rangle|0\rangle,\ |2\rangle|3\rangle|3\rangle\}\}. \tag{D8}$$

Evidently, these two sets are LOCC distinguishable. If M_3' clicks, the given state is one of the 13 remaining states.

Step 5. Alice performs the measurement

$$\mathcal{K}' \equiv \{K_1' := \mathbb{P}[|1\rangle_A; \mathbb{I}_a], K_2' := \mathbb{P}[|2\rangle_A; \mathbb{I}_a], K_3' := \mathbb{I} - K_1' - K_2'\}.$$

States corresponding to the outcomes are

$$\begin{aligned}\{K_1' &\Rightarrow \{|1\rangle|1\rangle|3\rangle,\ |1\rangle|1\rangle|0\rangle,\ |1\rangle|1\rangle|1\rangle\},\\ K_2' &\Rightarrow \{|2\rangle|1\rangle|0\rangle,\ |2\rangle|1\rangle|1\rangle,\ |2\rangle|2\rangle|0\rangle,\ |2\rangle_A(|1\rangle_B|000\rangle_{abc} \pm |2\rangle_B|111\rangle_{abc}\pm)|3\rangle_C\},\\ K_3' &\Rightarrow \{|3\rangle|1\rangle|1\rangle,\ |3\rangle|1\rangle|3\rangle,\ |3\rangle|2\rangle|3\rangle,\ |3\rangle_A(|1\rangle_B|000\rangle_{abc} \pm |2\rangle_B|111\rangle_{abc}\pm)|0\rangle_C\}\}.\end{aligned} \tag{D9}$$

Evidently, all three of these sets are LOCC distinguishable.

Remark 1. If N_2 clicks in step 3, then the given state is one of

$$\{|3\rangle|2\rangle|\xi_\pm\rangle,\ |2\rangle|\chi_\pm\rangle|2\rangle,\ |\chi_\pm\rangle|3\rangle|1\rangle,\ |3\rangle|3\rangle|2\rangle,\ |2\rangle|2\rangle|1\rangle\}. \tag{D10}$$

Considering the relabeling $2 \mapsto 1$ and $3 \mapsto 0$ for Alice and Bob and $1 \mapsto 1$ and $2 \mapsto 0$ for Charlie, the above set reads

$$\{|0\rangle|1\rangle|0 \pm 1\rangle,\ |1\rangle|0 \pm 1\rangle|0\rangle,\ |0 \pm 1\rangle|0\rangle|1\rangle,\ |0\rangle|0\rangle|0\rangle,\ |1\rangle|1\rangle|1\rangle\}. \tag{D11}$$

It is the OPB corresponding to the shift UPB of $(\mathbb{C}^2)^{\otimes 3}$ [6], and this set can be perfectly distinguished under LOCC if a two-qubit maximally entangled state is shared between any two parties. Since the unknown state is chosen at random (i.e., with uniform probability) from the set $\mathbb{B}_{\text{II}}(4, 3)$, the total entanglement consumed in this protocol is one GHZ state and $\frac{1}{8}$ of an EPR state.

Remark 2. Instead of a GHZ resource, consider that Alice and Bob share a two-qubit maximally entangled state. After step 1, the tag is shared between Alice and Bob only. As already discussed in step 2, Alice can discriminate 20 states corresponding to the outcomes K_1 and K_2. However, if the K_3 outcome occurs, the discrimination protocol cannot proceed further if no more entangled resource is used. However, if an entangled state $|\phi^+\rangle_{b'c}$ is provided between Bob and Charlie then a perfect discrimination protocol is possible. For that Bob starts with a twist-breaking measurement $\{\mathbb{P}[(|1\rangle, |2\rangle)_B; |0\rangle_{b'}] + \mathbb{P}[(|0\rangle, |3\rangle)_B; |1\rangle_{b'}],\ \mathbb{I} - \mathbb{P}\}$. Then an analogous protocol follows that discriminates all 44 remaining states. Therefore, the total entanglement consumption in this protocol is $\{(1, |\phi^+\rangle_{\mathcal{AB}}); (\frac{11}{16}, |\phi^+\rangle_{\mathcal{BC}})\}$.

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