A Note on the Lower Bound of Black Hole Area Change in Tunneling Formalism

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Abstract

In the framework of tunneling mechanism and employing Bekenstein's general expression for the variation of the black hole area, we determine the area quantum up to a constant. Depending on the value of this constant one can get either Bekenstein's lower bound or Hod's one for the change in the black hole area.

Bekenstein was the first to show that there is a lower bound (quantum) in the increase of the area of the black hole horizon when a neutral (test) particle is absorbed [1]

$$(\Delta A)_{min} = 8\pi l_{pl}^2 \tag{1}$$

where $l_{pl} = (G\hbar/c^3)^{1/2}$ is the Planck length which, since we use gravitational units, i.e. $G = c = 1$, takes the form $l_{pl}^2 = \hbar$. Later on, Hod considered the case of a charged particle absorbed by a Reissner-Nordström black hole and derived a smaller bound for the increase of the black hole area [2]

$$(\Delta A)_{min} = 4l_{pl}^2 \ . \tag{2}$$

At the same time Hod put forward a diverse proposal for the computation of the quantum of the black hole horizon area. Hod's proposal combined the quasinormal modes of the perturbed astrophysical black holes with the principles of Quantum Mechanics and Statistical Physics [3]. Motivated by this proposal, Kunstatter employed the idea of adiabatic invariants and the statement by Bekenstein [4] in order to derive for the $d(\geq 4)$-dimensional Schwarzschild black hole an equally spaced entropy spectrum [5]. Kunstater's computation was based on the first law of black hole mechanics, the Bohr-Sommerfeld quantization condition, and Hod's proposal. The question raised by Kunstatter was if his computation could be extended to the rotating black holes. At that time, the quasinormal frequencies of the rotating black holes hadn't been computed. Berti, Cardoso, and Yoshida were the first to evaluate the quasinormal modes of Kerr black holes using numerical methods [6]. Later on their results were confirmed by analytical works [7, 8].

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Almost ten years after Hod's proposal and immediately following [7, 8], Maggiore gave a new interpretation for the black hole quasinormal modes which rejuvenated the interest in this direction [9]. In this context the area spectrum is evenly spaced and the area quantum for the Schwarzschild as well as for the Kerr black hole is given by equation (1) [10, 11]. While this is in agreement with the old result of Bekenstein, it completely disagrees with Hod's lower bound as given by equation (2).

Very recently, utilizing completely different arguments, Ropotenko [12, 13] as well as Medved [14] showed that the minimum bound on the black hole area spectrum is given by (1).

However, at the same time, we have computed the black hole area spectrum employing tunneling mechanism [15, 16] and proved that the area quantum is identical to the minimum bound (2).
It is evident that there is a factor of 2π discrepancy between the recent results for the area spectrum; while some give support to Bekenstein's result others find Hod's result. Therefore, the main question arises whether it is possible to simultaneously obtain both the existing results in literature.

In this paper, we shall show that this is feasible when the tunneling mechanism is employed. Describing in brief the way that this can be materialized, one has to compute the uncertainty in the energy of the emitted particle from the black hole. Then, using this uncertainty in conjunction with Heisenberg's uncertainty relation in Bekenstein's general expression for the variation in area [17], one obtains the minimum change in the black hole area. This final expression will contain an undetermined constant. Specific choices on the value of this constant will lead either to Bekenstein's area quantum as given by equation (1) or to Hod's quantum area given by equation (2). Finally, it will be shown that exploiting the first law of black hole mechanics, instead of Bekenstein's general expression for the change in area, one obtains a minimum of the area change which is identical to Hod's result.

It is well known that near the horizon the theory is dimensionally reduced to a 2-dimensional theory [18, 19] whose metric is just the $(t-r)$ sector of the original metric while the angular part is red shifted away. Consequently the near horizon metric has the form,

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} \ . \tag{3}$$

The horizon is defined by the relation $F(r = r_H) = 0$ and the surface gravity is given by $\kappa = \frac{F'(r_H)}{2}$. Now consider the massless Klein-Gordon equation $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0$ under the metric given in equation (3)

$$-\frac{1}{F(r)}\partial_t^2\phi + F^{'}(r)\partial_r\phi + F(r)\partial_r^2\phi = 0 \ . \tag{4}$$

Taking the standard WKB ansatz $\phi(r,t) = e^{-\frac{i}{\hbar}S(r,t)}$ and substituting the expansion for $S(r,t)$

$$S(r,t) = S_0(r,t) + \sum_{i=1}^{\infty}\hbar^i S_i(r,t) \tag{5}$$

in equation (4) we obtain the solutions for ϕ in the semiclassical limit, i.e. $\hbar \to 0$, [21, 22]

$$\phi^{(R)}_{in} = e^{-\frac{i}{\hbar}\omega u_{in}} \quad ; \qquad \phi^{(L)}_{in} = e^{-\frac{i}{\hbar}\omega v_{in}} \tag{6}$$

$$\phi^{(R)}_{out} = e^{-\frac{i}{\hbar}\omega u_{out}} \ ; \qquad \phi^{(L)}_{out} = e^{-\frac{i}{\hbar}\omega v_{out}} \tag{7}$$

where the quantity ω is the energy of the particle as measured by an asymptotic observer. Here "R (L)" refers to the outgoing (ingoing) mode while "in (out)" stands for inside (outside) the event horizon. The null coordinates (u, v) are defined as

$$u = t - r^* \ , \qquad v = t + r^* \ ; \qquad dr^* = \frac{dr}{F(r)} \ . \tag{8}$$

In the context of the tunneling formalism, a virtual pair of particles is produced in the black hole. One member of this pair can quantum mechanically tunnel through the horizon. This particle is observed at infinity while the other goes towards the center of the black hole. While crossing the horizon the nature of the coordinates changes. This can be accounted by working with Kruskal coordinates which are viable in both sectors of the black hole event horizon. The Kruskal time (T) and space (X) coordinates inside and outside the horizon are defined as [20]

$$T_{in} = e^{\kappa r^*_{in}} \cosh(\kappa t_{in}) \quad ; \qquad X_{in} = e^{\kappa r^*_{in}} \sinh(\kappa t_{in}) \tag{9}$$

$$T_{out} = e^{\kappa r^*_{out}} \sinh(\kappa t_{out}) \quad ; \qquad X_{out} = e^{\kappa r^*_{out}} \cosh(\kappa t_{out}) \ . \tag{10}$$

These two sets of coordinates are connected through the following relations

$$t_{in} = t_{out} - i\frac{\pi}{2\kappa} \tag{11}$$

$$r^*_{in} = r^*_{out} + i\frac{\pi}{2\kappa} \tag{12}$$

so that the Kruskal coordinates get identified as $T_{in} = T_{out}$ and $X_{in} = X_{out}$. Employing equations (11) and (12) in equation (8), we can obtain the relations that connect the radial null coordinates defined inside and outside the black hole event horizon

$$u_{in} = t_{in} - r^*_{in} = u_{out} - i\frac{\pi}{\kappa} \tag{13}$$

$$v_{in} = t_{in} + r^*_{in} = v_{out} \ . \tag{14}$$

Under these transformations the modes in equations (6) and (7) which are travelling in the "*in*" and "*out*" sectors of the black hole horizon are connected through the expressions

$$\phi^{(R)}_{in} = e^{-\frac{\pi\omega}{\hbar\kappa}} \phi^{(R)}_{out} \tag{15}$$

$$\phi^{(L)}_{in} = \phi^{(L)}_{out} \ . \tag{16}$$

Concentrating on the modes located inside the horizon, the L mode is trapped while the R mode tunnels through the horizon [21, 22]. The probability for the R mode to travel from the inside to the outside of the black hole, as measured by an external observer, is given as

$$P^{(R)} = \left|\phi^{(R)}_{in}\right|^2 = \left|e^{-\frac{\pi\omega}{\hbar\kappa}} \phi^{(R)}_{out}\right|^2 = e^{-\frac{2\pi\omega}{\hbar\kappa}} \tag{17}$$

where equation (15) has been used to extract the final expression. Since the measurement is done from the outside, $\phi^{(R)}_{in}$ has to be expressed in terms of $\phi^{(R)}_{out}$. Therefore the average value of the energy, measured from outside, is written as

$$< \omega > = \frac{\int_0^\infty \ d\omega \ \omega \ P^{(R)}}{\int_0^\infty \ d\omega \ P^{(R)}} = T_H \tag{18}$$

where $T_H = \frac{\hbar\kappa}{2\pi}$ is the Hawking temperature. In a similar way, one can compute the average squared energy of the particle, detected by an asymptotic observer,

$$< \omega^2 > = \frac{\int_0^\infty \ d\omega \ \omega^2 \ P^{(R)}}{\int_0^\infty \ d\omega \ P^{(R)}} = 2T_H^2 \ . \tag{19}$$

Hence it is straightforward to evaluate the uncertainty in the detected energy ω by combining equations (18) and (19),

$$(\Delta\omega) = \sqrt{<\omega^2> - <\omega>^2} = T_H \tag{20}$$

which is nothing but the Hawking temperature T_H.

Now according to Bekenstein [17], the change in area of a black hole caused either by an absorption or by an emission of a particle is given by

$$\Delta A \geq 8\pi \int_V x T_{00} dV \tag{21}$$

where x is the distance of the center of mass of the particle from the horizon and T_{00} represents the energy density corresponding to the particle. Here V stands for the volume (a 3-surface) of the system, i.e. the black hole and the particle, outside the black hole, at a constant time. Based on dimensional grounds, we can consider the position x of the emitted particle to be of the order of the uncertainty in particle's position, i.e. (ΔX). Then one can set $x = \epsilon \Delta X$[1], where ϵ is some constant that fixes the equality. Therefore, equation (21) can be written as

$$\Delta A \geq 8\pi\epsilon \ \Delta X \ \int_V T_{00} dV \ . \tag{22}$$

It is evident that the value of the integration on the right hand side of equation (22) is exactly the energy of the outgoing particle. Since the integration is performed at a constant time over the whole space outside the black hole, it is legitimate to identify the energy of the particle as computed through the integration with the average energy of the particle given by equation (18). Substituting this in the above equation we obtain

$$\Delta A \geq 8\pi\epsilon \ \Delta X \ T_H \tag{23}$$

where the uncertainty (ΔX) has to be determined.
For a massless particle its momentum p is defined as $p = \omega$ (with $c = 1$). Therefore, the uncertainty in the particle's momentum, i.e. Δp, is equal to the uncertainty in particle's energy, i.e. $\Delta\omega$, as given by equation (20). Now, implementing the Heisenberg uncertainty relation $\Delta X \ \Delta p \ \geq \hbar$ and substituting equation (20), the uncertainty in particle's position reads

$$\Delta X \geq \frac{\hbar}{\Delta p} = \frac{\hbar}{\Delta\omega} = \frac{\hbar}{T_H} \tag{24}$$

and thus

$$8\pi \Delta X T_H \geq 8\pi\hbar \ . \tag{25}$$

Substituting equation (25) in equation (23), the inequality that the change in the area of the black hole satisfies, takes the form

$$\Delta A \geq 8\pi\epsilon l_{pl}^2 \ . \tag{26}$$

It is straightforward that the minimum value of the change in the black hole area is given as

$$(\Delta A)_{min} = 8\pi\epsilon l_{pl}^2 \ . \tag{27}$$

It should be stressed that for different values of the constant ϵ, one can get the existing values in the literature. In particular, if $\epsilon = 1$ then Bekenstein's result given by equation (1) is recovered while if $\epsilon = 1/2\pi$, Hod's result as described by equation (2) is obtained.

[1] It is noteworthy that distance x as well as the uncertainty in particle's position ΔX are treated as measured in the rest frame of the system. Due to length contraction, there should be a Lorentz factor to the measured quantities due to the relative motion between the particle (in particular, of its center of mass) and the frame of observation. This makes possible to have values for ϵ less than unity.

For the sake of completeness, we want to mention that the change in the black hole area can also be discussed by using the first law of black hole mechanics [15, 16], instead of using equation (21). In this framework, considering the uncertainty in the energy of the particle detected by an asymptotic observer (see equation (20)) as the change of energy in the first law of black hole mechanics, we get

$$T_H \Delta S_{bh} = \Delta\omega \tag{28}$$

and thus the entropy change takes the form

$$\Delta S_{bh} = 1 \ . \tag{29}$$

Since the entropy of a black hole in *Einstein* theory is given by the Bekenstein-Hawking formula

$$S_{bh} = \frac{A}{4l_{pl}^2} \ , \tag{30}$$

it is clear that the change in the black hole area is now [15],

$$\Delta A = 4l_{pl}^2 \ . \tag{31}$$

This agrees with Hod's result [2] for the minimum change in black hole area.

To conclude, in the first approach that we adopted here both the existing results in the literature are reproduced, whereas the second approach fails to give Bekenstein's result. In addition, this later approach leads directly to an equality, rather than to an inequality which the change in black hole area must satisfy. Hence the concept of area quantum through the minimization of the change in the black hole area does not arise. The main reason is that in the first case we have considered equation (21) which corresponds to Hawking's area theorem in which an inequality holds, whereas the first law of black hole mechanics (see equation (28)) which is taken as an input in the second approach, is an equality.
Finally, it should be stressed that our approach is based on the near-horizon mode solutions (6) and (7) which are plane waves. In this region the effective potential vanishes and there are no grey-body factors. However, the self consistency of the approach can be seen by recalling that the emission spectrum obtained from these modes is purely thermal (see for details [22]). This justifies ignoring the grey-body factors.

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