Externally driven granular systems

S.S. Manna*

Satyendra Nath Bose National Centre for Basic Sciences, Block-JD, Sector-III, Salt Lake, Calcutta 700 091, India

Abstract

A system of granular material resting in a bin is driven by creating voids in the bulk of the medium. Modeling the grains by a collection of mono-disperse discs, we simulate the two dimensional version of the system. Using a 'pseudo dynamics' to study the dynamics of the system, we observe that the system reaches a steady state in long times. We study different properties of the steady state. *PACS:* 05.70.Jk; 64.60 Lx; 74.80.Bj; 46.10.+z

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1. Introduction

In a granular medium at rest, different grains are in equilibrium under mutually acting balanced contact forces. Now, if a grain is moved, locally the force balance is disturbed. How far does this disturbance propagate in the medium? This is the main question we are trying to discuss in this paper.

Our system consists of a granular material resting in an open bin. Grains, one at a time, residing at the bottom of the bin are randomly selected and removed from the system. We call this process as the external perturbation. Every time a grain is removed, a cascade of grain displacements propagates in the interior of the granular material, which is called an 'avalanche'. After the termination of the avalanche, the removed grain is placed back randomly on the top of the system, to conserve the total number of grains in the system. The next avalanche is created by removing again a grain at the bottom.

Since the system has an external driving mechanism (repeated void creations), and a dissipation mechanism (cascade of grain displacements) involving two widely different

* Corresponding author. Tel.: +91-33-335-5708; fax: +91-33-335-3477.
E-mail address: manna@boson.bose.res.in (S.S. Manna)

time scales (a void is created only after the termination of the previous avalanche), it fulfills all the criteria of a self-organized critical system (SOC) [1,7]. Therefore, we expect that our granular system, starting from any arbitrary initial packing pattern, would reach a steady state after a long time, where long range correlations will be developed, the signature of which will be manifested in the power law distributions of the avalanche sizes.

This problem was first studied in [2] using a semi-lattice model. Grains were modeled by unit squares, whose positions could vary continuously along the horizontal direction, where as the vertical coordinates were discretized. Later publications in [3–5] studied this problem with discs as well as using cellular automata. SOC was observed in all these models.

2. Model

We study the two dimensional version of this system. The granular bin is represented by a rectangular area on the x–y plane: from $x = 0$ to L_x and $y = 0$ to L_y. Periodic boundary condition is imposed along the x direction and gravity acts along the $-y$ direction. The bottom of the bin coinciding with $y = 0$ line is highly sticky and any grain which comes in its contact gets stuck there and does not move further.

The granular material is a collection of grains modeled by N hard mono-disperse discs of radii R. No two grains are allowed to overlap but can come very close to each other. In fact a grain can touch another grain and roll on it.

The initial grain pattern is generated by the 'ballistic deposition and restructuring method (BDRM)'. In this method, grains are released at random positions at the top level of $y = L_y$ sequentially one after the other. Subsequently, they are allowed to fall vertically till they come in contact with the pile when they roll down to their stable positions along the paths of steepest descent.

When all N grains are dropped, we have the initial configuration. A list of the discs residing at the bottom is made. To create an avalanche, an arbitrary disc at the bottom is chosen and is deleted. The neighbouring discs which were residing on it, become unstable and start moving. Consequently, the discs in the further neighbourhood also move. This cascade of grain movements is an avalanche, which finally stops.

3. Algorithm

Each grain is assigned a serial number n from 1 to N. The centre coordinates of these grains are stored in the $x_N(n)$ and $y_N(n)$ arrays. Very often we search the set of grains that reside in the local neighbourhood of a particular grain n. As a brute force solution one can calculate the distances r_{nm} of the centres of all other grains $(m = 1, N; m \neq n)$ from the centre of the nth grain and pick up those grains which are within the local neighbourhood. This takes CPU proportional to N.

The search is done more efficiently by considering an underlying grid $S(0:L_x, 0:L_y)$. A primitive cell defined by the area extending from $x = i$ to $i+1$ and $y = j$ to $j+1$, where i and j are the integer numbers. The cell is referred by the coordinates of its bottom-left corner (i, j). If the centre of a grain n with coordinates (x_n, y_n) is within the primitive cell (i, j) then the location (i, j) in the S array is assigned the grain number n. We ensure that any primitive cell can contain the centre of at most one grain by a suitable choice of the radii of the grains $R = 1/\sqrt{2} + \varepsilon$, where ε is a small positive number. Therefore if the $S(i, j)$ location is zero, it implies that the cell (i, j) does not contain the centre of any grain.

In our simulation we define the local neighbourhood LN of a grain whose centre is within the cell (i, j) by the area extending from ($x = i - 2$ to $x < i + 3$ and $y = j - 2$ to $y < j + 3$). Only the grains, whose centres are with LN may overlap with the grain whose centre is within the cell (i, j). While searching for the set of grains whose centres are within LN, we search the lattice locations $i - 2$ to $i + 2$ and $j - 2$ to $j + 2$. Numbers stored in these locations give the grain numbers and their centre coordinates are obtained from the $x_N(n)$ and $y_N(n)$ arrays. This search takes CPU proportional to N^0.

The temporal evolution of the granular system is studied by a 'pseudo-dynamics'. Unlike the method of molecular dynamics we do not solve here the classical equations of motion for the grain system. Only the direction of gravity and the local geometrical constrains due to the presence of other grains govern the movement of a grain. In our method a grain can have only two possible movements namely the un-obstructed vertical *fall* and the un-obstructed *roll* over another grain in contact. A parameter δ is introduced to characterize the fall and the roll moves. Time is discretized and in a unit time a grain may fall to a maximum height of δ or its centre can roll a maximum angle of $\theta = \delta/2R$ over another grain. During both fall and roll a grain does not accelerate.

The initial grain pattern is generated by the BDRM method. In this method grains are released at random positions in the top level of $y = L_y$ sequentially one after the other, and are allowed to fall vertically till they come in contact with the pile when they roll to their stable positions through successive rolls on the surface of the granular heap (Fig. 1).

Since the grains are all of equal radii, a grain may be in contact with a maximum of six other grains. To recall quickly the set of grains in contact with an arbitrarily selected grain n we store the serial numbers of the contact grains for every grain into a two dimensional array $neb(N, 6)$. The grain n may be at rest on two other grains and its weight is balanced by the reaction forces from them. The centres of the supporting grains must be on the two opposite sites of the vertical line passing through the centre of the stable grain n. We reserve the location $neb(n, 1)$ to store the serial number n_L of the left supporting grain and $neb(n, 2)$ to store the serial number n_R of the right supporting grain. The serial numbers of all other grains which are in contact with the grain n are kept one after the other in the locations $neb(n, m)$, $m = 3, 6$.

To find out the neighbours in contact with a grain n we follow the following procedure. We first make a list of grains whose centres are within the local neighbourhood

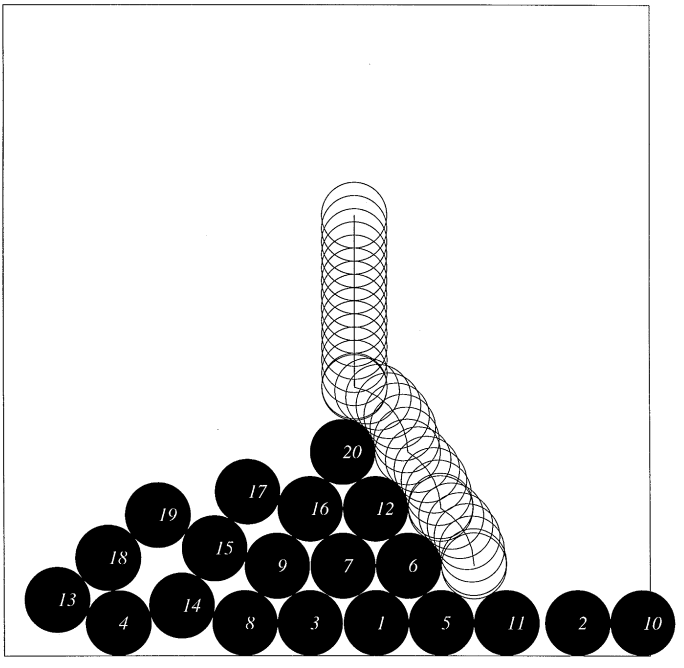

Fig. 1. Procedure for adding a single grain to the surface of a granular heap in a bin, using the Ballistic deposition by restructuring method. The already formed heap has 20 grains. The 21st grain is being deposited in a sequence of successive 'fall' and 'roll' moves.

LN of the grain n. From this list, we sort out those grains whose centres are within a distance $2R - \varepsilon$ to $2R + \varepsilon$ from the centre of n, ε being the tolerance factor which takes care of the accumulation of errors originated from real number manipulations. We consider these grains are in contact to the grain n. For each contact grain we calculate the angle ϕ measured from the vertically downward direction through (x_n, y_n) to the vector starting from (x_n, y_n) to the centre of the contact grain. This angle ϕ is measured $+$ve in the anti-clockwise direction and $-$ve in the clockwise direction. We first find out two contact grains with minimum values of ϕ in the positive and the negative directions. If the sum of the magnitudes of these two angles is less than π, the grain n is considered at rest.

When a grain n is updated, it first selects the type of movement it is going to take:

- If $n_L = n_R = 0$ the grain n is allowed to fall.
- If $n_L \neq 0$ but $n_R = 0$ the grain n is allowed to roll on the right over the supporting grain n_L in contact at the left.
- If $n_L = 0$ but $n_R \neq 0$ the grain n is allowed to roll on the left over the supporting grain n_R in contact at the right.
- If $n_L \neq 0$ and $n_R \neq 0$ the grain n is considered a stable and does not move.

The $fall$ movement of a grain n is executed in the following way. Corresponding to every grain in the local neighbourhood LN we first calculate the distance through

which n should climb down vertically to make a contact. The minimum δ_m of these distances corresponds to the grain r with the centre at (x_r, y_r). If $\delta_m < \delta$ the grain n is brought down a distance δ_m so that it just touches r. The new coordinates are given by:

$$x'_n = x_n, \qquad y'_n = y_r + \sqrt{4R^2 - (x_n - x_r)^2}\,.$$

However if $\delta_m \geqslant \delta$, the grain n falls a distance δ only. The neighbour lists of the grains n and r are updated using the subroutine *sub_neb*. The lattice S is updated for the movement of the grain n and therefore the lattice point corresponding to (x_n, y_n) is vacated whereas that corresponding to (x'_n, y'_n) is occupied.

Now consider the situation when a grain n *rolls* over another grain r with the centre at (x_r, y_r). The minimum angle θ_m through which the grain n should freely roll to become in touch with another grain t in the neighbourhood LN with the centre at (x_t, y_t) is calculated. If $\theta_m < \theta$ the grain n is rolled an angle θ_m over the grain r so that it simultaneously touches both the grains r and t. The centre coordinates (x'_n, y'_n) of the grain n in the new position are given by

$$x'_n = \frac{1}{2}(x_t + x_r) + g(y_t - y_r)\sqrt{\frac{4R^2}{d_{rt}^2} - \frac{1}{4}}\,,$$

$$y'_n = \frac{1}{2}(y_t + y_r) - g(x_t - x_r)\sqrt{\frac{4R^2}{d_{rt}^2} - \frac{1}{4}}\,,$$

where d_{rt} is the distance between the centres of the grains r and t. If the angle $\theta_m \geqslant \theta$, the grain n rolls an angle θ over the grain r. In both cases if it happens that $y'_n < y_n$, then the grain n is brought at the same level as the grain r and we make $x'_n = x_n + 2gR$ and $y'_n = y_n$. The lattice S is renewed for the movement of the grain n and therefore the lattice point corresponding to (x_n, y_n) is vacated whereas that corresponding to (x'_n, y'_n) is occupied. The factor g is $+1$ while the grain n is rolling on the left and it is -1 while n rolls on the right. The neighbour lists of the grains n and t are also updated.

The system of N grains resting in the bin are disturbed by selecting an arbitrary grain n at the bottom of the bin and removing it from the system at time $t = 0$. A number of neighbouring grains depend on the grain n for their stability. These grains will be unstable and tend to move. In general, at any time t we have a list of unstable grains. When we update them some become stable but others remain unstable. Our new list for the time $t + 1$ consists of these unstable discs from time t plus the neighbouring grains which become unstable due to motion of grains in time t. The avalanche of grain displacement stops when the list is empty. In a certain time t the grains are updated sequentially.

After the avalanche stops, the removed grain is kept back at the surface of the pile by releasing again at an arbitrary position at the top level.

Our simulation code consists of five subroutines along with the main code. A subroutine *sub_fall*(n, i, j) finds out the position of the disc n after it makes a free fall to a maximum distance of δ. Secondly, the subroutine *sub_leftroll*(n, i, j) calculates the

position of the disc n after it rolls to the right over a disc in contact at the left. The subroutine $sub\_rightroll(n,i,j)$ calculates the position of the disc n after it rolls to the left over a disc in contact at the right. The subroutine $sub\_neb(n)$ calculates the contact neighbourhood of the grain number n and returns after properly updating the six locations of $neb(n,k)$, $k=1,6$. This subroutine is called after every successful move of the grain n. Then the neighbour list of every grain in contact of n is updated by calling again this subroutine. The last subroutine is called $sub\_rsf(n,x_N(n),y_N(n))$. Here the grain n has the current values for the coordinates of its centre as $(x_N(n),y_N(n))$ and from this point it is allowed to relax to its stable state on the surface of the heap by a succession of falls and rolls.

In the main code we start with parameters for the bin size L, grain radius R and the number of grains N. All N grains are sequentially dropped from the level $y=L_y$. For every grain the subroutine $sub\_rsf(n,x_N(n),y_N(n))$ is called and the grain is taken to its local stable position on the top of the pile already formed. This gives us the initial grain pattern. We also call this subroutine while replacing the removed grain randomly on the top of the surface.

The avalanche is created by removing an arbitrary grain from the bottom. During the progress of the avalanche when we update a grain n we first check its first two locations in the neb array i.e., $neb(n,1)$ and $neb(n,2)$ and determine which type of movement it is going to execute. If both locations contain non-zero numbers then the grain is stable and we pass on to update the next grain.

4. Results

The area coverage of the packing pattern in the steady state is found to vary with the vertical height. We measure the area coverage $\rho(y)$, averaged over an unit strip between y and $y+1$ by the amount of area covered by the discs. In Fig. 2, we plot $\rho(y)$ for three different system sizes of 2000, 4000 and 8000 grains contained in a bin of size $L_x=100$. We observe that the pattern become denser with increasing y. We explain that most of the arches reside at the bottom creating more vacant regions at the bottom. There are many avalanches which do not reach the surface, therefore fewer arches are formed with increasing height and packing becomes more compact. We expect that the area coverage should grow to a maximum of 0.8180 ± 0.0002 as quoted in [6] for the initial configuration.

In a stable configuration, the grains residing at the bottom support the weights of the grains above. A grain at the bottom supports the partial weights of the few other neighbouring contact grains residing above it. These neighbouring grains also supports further neighbouring grains. Therefore, each grain at the bottom supports a large number of grains above it. We call the set of supporting grains as the 'supporting cluster' corresponding to the grain at the bottom (Fig. 3).

We measure the size of the supporting cluster s_s as the number of supported grains. While creating an avalanche, we randomly select a grain at the bottom. Before deleting

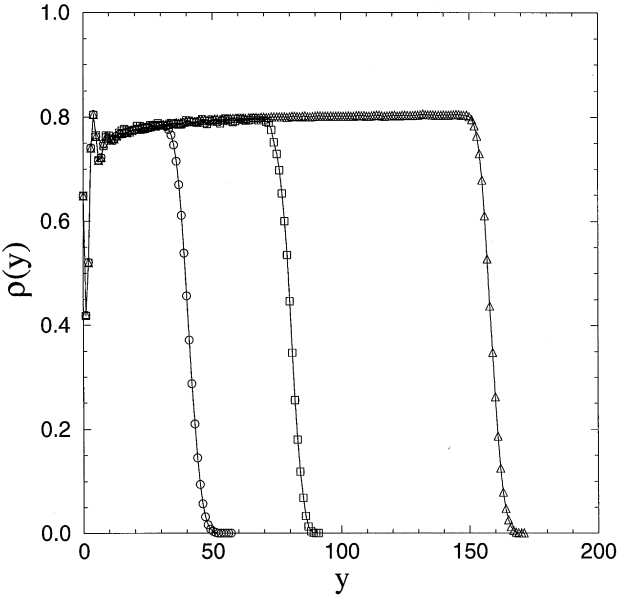

Fig. 2. The local average area coverage $\rho(y)$ is plotted with the vertical height y for the systems of 2000 (circle), 4000 (square) and 8000 (triangle) grains in a bin of width 100.

it, we calculate the value of (s_s). The supporting cluster distribution is plotted using a double logarithmic scale in Fig. 4 for three system sizes of 2000, 4000 and 8000 grains contained within a bin of size $L_x = 100$. We see that each plot is having two distinctly separate regions. For small values of s_s we see a sharply decreasing variation, far from being a power law. For large values of s_s we see a peaked variation. We explain, that in most cases, the supporting cluster reaches the surface of the heap. However, in some cases, the supporting cluster is completely surrounded by an arch and therefore the cluster cannot be extended to the surface. Since in the steady state, the arches are not of all length scales, and have a characteristic size, the supporting clusters which do not reach the surface also has a characteristic size.

The avalanche cluster size s is measured by the total number of grains displaced in an avalanche in comparison with the initial configuration. A snapshot of the configuration showing the supporting cluster, avalanche cluster as well as the set of undisturbed grains are shown in Fig. 3. The avalanche cluster size distribution is also plotted using a double logarithmic scale in Fig. 5. For small values of s we get a value of the slope around 2.95. If we assume that the distribution is described by a power law for small s, as $D(s) \sim s^{-\tau}$, then $\tau = 2.95$.

Now we comment on the effect of the parameter δ. Since, during the avalanche, different grains are updated in a sequence, the final steady configuration depends on the sequence in which different grains were updated. In the limit of $\delta \to 0$, this dependence will vanish. We observe that, on decreasing δ, the avalance cluster size

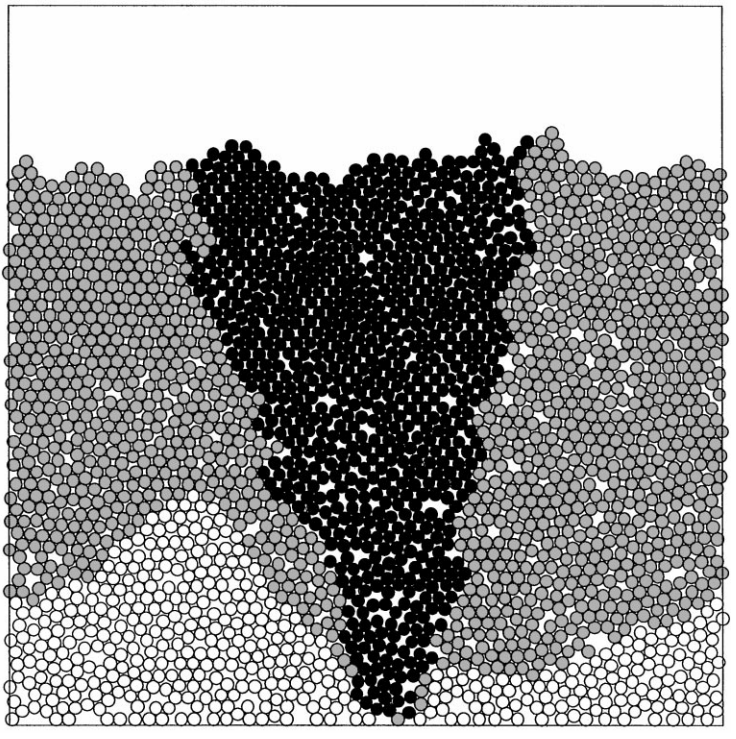

Fig. 3. The steady state configuration for a system of 2500 grains in a bin of $L_x = 80$. The gray and the black circles denote the grains in the supporting cluster of the lower most circle at the bottom. If this circle is deleted, only the black circles will move, which constitute the avalanche cluster. The vacant circles are the undisturbed grains.

increases and in the limit of $\delta \to 0$, the avalanche cluster and the supporting cluster should be the same.

Therefore, we believe that the exponent $\tau = 2.95$ is actually a result of the finite $\delta = R/10$ used in our simulation. If we reduce delta, and take it to the limit of $\delta \to 0$, the $D(s)$ vs. s plot should look alike as the $D(s_s)$ vs. s_s plot. To show this numerically, however, turned out to be difficult. We simulated a particular cluster in a system of N=2500, L_x=80, s_s=2068, using $\delta, 10^{-1}\delta, 10^{-2}\delta$. The avalanche cluster sizes obtained are: 1046, 1080, 1081 respectively.

We conclude that, small avalanches, which donot reach the granular surface are unlikely to be of all sizes because the lengths of the arches present in the system in the stationary state are not of all sizes. Therefore an arbitrary avalanche is most likely to reach the surface. However, the avalanche size has a distribution, which looks like a power law for finite δ but we expect that this behaviour is temporary and in the limit of $\delta \to 0$, the distribution should fall faster like that of the supporting cluster size distribution.

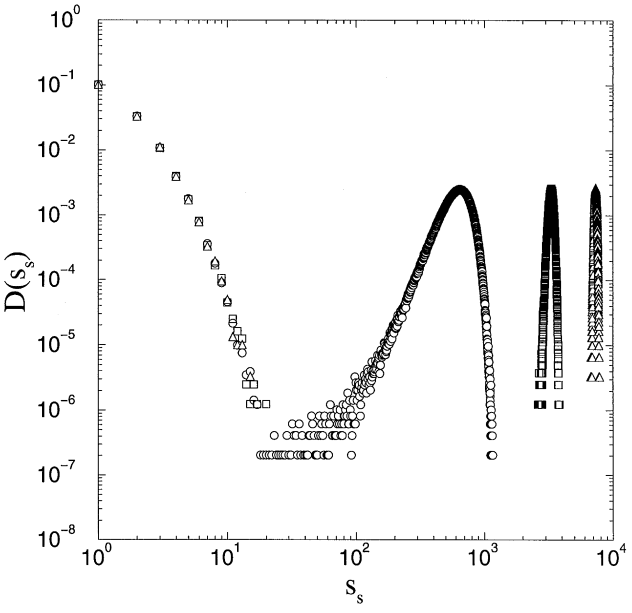

Fig. 4. Probability distributions of the sizes of the supporting clusters (s_s) for the systems of 2000 (circle), 4000 (square) and 8000 (triangle) grains in a bin of width 100. Each curve has two parts. For small vales of s_s, all curves fall on one another. For the large (s_s) values, the curve has a peak, and the peak shifts linearly with the average height of the heap.

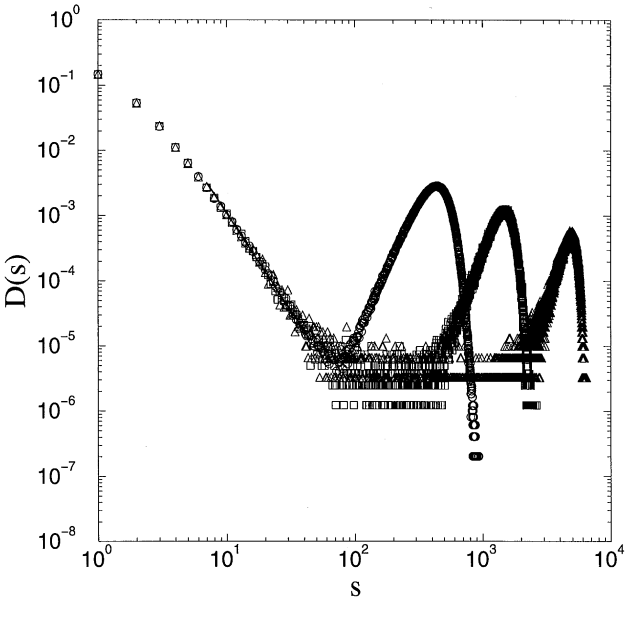

Fig. 5. Probability distributions of the sizes of the avalanche clusters (s) for the same systems as in Fig. 4.

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