

Master equation for lagrangian gauge symmetries

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Abstract

Using purely Hamiltonian methods we derive a simple differential equation for the generator of the most general local symmetry transformation of a Lagrangian. The restrictions on the gauge parameters found by earlier approaches are easily reproduced from this equation. We also discuss the connection with the purely Lagrangian approach. The general considerations are applied to the Yang-Mills theory.

The problem of finding the most general local symmetries of a Lagrangian has been pursued by various authors, using either Lagrangian [1–4] or Hamiltonian techniques [5–7]. Nevertheless a compact equation which determines the precise structure of the generator of gauge transformations, which are the symmetries of a Lagrangian, is still lacking.

In a recent paper [8] we had shown that the requirement of commutativity of the time derivative operation with an arbitrary infinitesimal gauge transformation generated by the first class constraints was the only input needed for obtaining the restrictions on the gauge parameters entering the most general form of the generator G . The analysis was performed entirely in the Hamiltonian framework.

Following essentially the same commutativity requirement, we derive here, in a Hamiltonian frame-

work, a simple differential equation for the generator. This differential equation encodes, in particular, the restrictions on the gauge parameters. We explicitly demonstrate that this equation implies the (off shell) invariance of the action under the transformation generated by G , and ensures the covariance of the Hamilton equations of motion.

In this paper we shall consider purely first class systems. The extension to mixed first and second class systems is straightforward. To keep the algebra simple, and also for reasons of comparison, we assume all constraints to be irreducible.

We consider a Hamiltonian system whose dynamics is described by the total Hamiltonian

$$H_T = H_c + \sum_{a_1} v^{a_1} \Phi_{a_1}, \quad (1)$$

where H_c is the canonical Hamiltonian, $\{\Phi_{a_1} \approx 0\}$ are the (first class) primary constraints, and v^{a_1} are the associated Lagrange multipliers. We denote the complete set of (primary and secondary) constraints ⁴ by

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⁴ “Secondary” refers to all generations of constraints beyond the primary one.

$\{\Phi_a\} = \{\Phi_{a_1}, \Phi_{a_2}\}$. The Poisson algebra of the constraints with themselves and with the canonical Hamiltonian, is of the form

$$[H_c, \Phi_a] = V_a^b \Phi_b, \quad (2)$$

$$[\Phi_a, \Phi_b] = C_{ab}^c \Phi_c, \quad (3)$$

where V_a^b and C_{ab}^c may be functions of the phase-space variables. Consider an infinitesimal transformation on the coordinates generated by G ,

$$\delta q^i = [q^i, G], \quad (4)$$

with

$$G = \sum_a \epsilon^a \Phi_a, \quad (5)$$

where, following Dirac's conjecture [9], the sum includes all of the first class constraints. The gauge parameters are allowed to depend in general on time, as well as on the phase space variables and Lagrange multipliers.

In Eq. (4) we have chosen to include the gauge parameters ϵ^a inside the bracket. The difference between Eq. (4) and the variation δq^i computed with the parameters outside the bracket is proportional to the constraints. Such terms can always be written in the form $\Lambda_{ij} \frac{\delta S}{\delta q_j}$, with $\Lambda_{ij} = -\Lambda_{ji}$, which correspond to trivial gauge transformations [5]. In this paper we are only considering gauge transformations, modulo these trivial gauge transformations.

Consider the first of Hamilton's equations, giving the connection between the velocities and the momenta

$$\frac{dq^i}{dt} \approx [q^i, H_T], \quad (6)$$

where \approx stands for "weak equality" in the sense of Dirac [9]. From above we obtain

$$\frac{d}{dt} \delta q^i \approx [[q^i, G], H_T] + \left[q^i, \frac{\partial}{\partial t} G \right], \quad (7)$$

and

$$\delta \frac{dq^i}{dt} \approx [[q^i, H_T], G], \quad (8)$$

where we have taken account of the fact that G will in general depend explicitly on the time. Implementing the commutativity requirement [8]

$$\frac{d}{dt} \delta q^i = \delta \frac{d}{dt} q^i \quad (9)$$

by equating the last two expressions, and using the Jacobi identity, we arrive at the condition

$$\left[q^i, \frac{\partial}{\partial t} G + [G, H_T] \right] \approx 0. \quad (10)$$

Using the ansatz (5) as well as the algebra given in Eqs. (2) and (3), it follows that $\frac{\partial}{\partial t} G + [G, H_T]$ is given by a linear combination of the first class constraints:

$$\frac{\partial}{\partial t} G + [G, H_T] = \sum_a \xi^a \Phi_a. \quad (11)$$

Substituting this expression into (10), we arrive at the condition

$$\xi^a \frac{\partial}{\partial p_i} \Phi_a \approx 0. \quad (12)$$

Now, the first class nature and linear independence (irreducibility) of the constraints guarantees that each of these can be identified as a momentum conjugate to some coordinate, the precise mapping being effected by a canonical transformation [2]. Since (12) holds for all i one is led to the condition $\xi^a \approx 0$. Therefore the r.h.s of Eq. (11) will be proportional to the square of the constraints, so that, within the Hamiltonian formalism, we are allowed to set the l.h.s of Eq. (11) strongly equal to zero:

$$\frac{\partial}{\partial t} G + [G, H_T] = 0. \quad (13)$$

This is the fundamental equation determining G , which we henceforth refer to as the "master equation". As we shall see, it will guarantee the (off-shell) invariance of the total action.

The condition (13) also ensures the covariance of the Hamilton equations of motion under a transformation generated by G . Thus consider the equation of motion for q^i :

$$\frac{dq^i}{dt} \approx [q^i, H_T(q, p, v)]. \quad (14)$$

Consider further the gauge-transformed phase space variables and Lagrange multipliers

$$\bar{q}^i = q^i + \delta q^i, \quad \bar{p}_i = p_i + \delta p_i, \quad \bar{v}^{a_1} = v^{a_1} + \delta v^{a_1},$$

with

$$\delta q^i = [q^i, G], \quad \delta p_i = [p_i, G],$$

$$\delta v^{a_1} = [v^{a_1}, G].$$

Using the equations of motion (14), the master equation (13) as well as (7) one readily verifies that

$$\frac{d\bar{q}_i}{dt} \approx [\bar{q}^i, H_T(\bar{q}, \bar{p}, \bar{v})],$$

which demonstrates the covariance. A similar statement holds for \bar{p}_i .

We now examine the implications of our condition (13) for the gauge parameters in Eq. (5). Making use of the algebra (2) and (3), the master equation (13) is easily seen to lead to

$$\begin{aligned} \frac{\partial}{\partial t} G + [G, H_T] &= \left(\frac{d\epsilon^b}{dt} - \epsilon^a [V_a^b + v^{a_1} C_{a_1 a}^b] \right) \Phi_b \\ &\quad - \delta v^{a_1} \Phi_{a_1} = 0. \end{aligned} \quad (15)$$

From here we obtain the following conditions:

$$\delta v^{b_1} = \frac{d\epsilon^{b_1}}{dt} - \epsilon^a [V_a^{b_1} + v^{a_1} C_{a_1 a}^{b_1}], \quad (16)$$

$$0 = \frac{d\epsilon^{b_2}}{dt} - \epsilon^a [V_a^{b_2} + v^{a_1} C_{a_1 a}^{b_2}]. \quad (17)$$

Note that in the above equations, $\frac{d\epsilon^a}{dt}$ denotes the total time derivative as given by

$$\frac{d\epsilon^a}{dt} = \frac{D\epsilon^a}{Dt} + [\epsilon^a, H_c] + v^{a_1} [\epsilon^a, \Phi_{a_1}], \quad (18)$$

where, following the notation of Ref. [5],

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{v}^{a_1} \frac{\partial}{\partial v^{a_1}} + \ddot{v}^{a_1} \frac{\partial}{\partial \dot{v}^{a_1}} + \dots \quad (19)$$

with an overdot denoting the derivative with respect to the explicit dependence in time. The same conditions have been obtained in Ref. [5] by looking at the invariance of the gauge-fixed *extended* action, and directly from the commutativity requirement (9) in Ref. [8]. Note that Eq. (16) only plays a role in the Hamiltonian formulation where the equations of motion are obtained from the variation of the total

action. For the extraction of the symmetries of the original Lagrangian the relevant equation is Eq. (17).

It is clear that the above considerations can be easily extended to the case where the dynamics is described by the *extended* Hamiltonian $H_E = H_T + v^{a_2} \Phi_{a_2}$. The commutativity requirement will now lead to the extended master equation

$$\frac{\partial}{\partial t} G + [G, H_E] = 0. \quad (20)$$

In this case no restrictions on the gauge parameters are implied by Eq. (20), which only determines the transformation law for the multipliers v^a :

$$\left(\frac{d\epsilon^b}{dt} - \epsilon^a [V_a^b + v^c C_{ca}^b] \right) - \delta v^b = 0. \quad (21)$$

This equation was obtained in Ref. [5] by directly looking at the invariance of the extended action. Furthermore, by imposing gauge conditions implementing $\{v^{a_2} = 0\}$ [5], one recovers Eqs. (16) and (17), as is evident from Eq. (20).

Returning to our formulation in terms of the total Hamiltonian, the first step for obtaining the final form for G consists in solving Eq. (17) for the ϵ^{a_2} 's in terms of the coordinates, momenta, Lagrange multipliers (including their time derivatives) and a set of independent parameters whose number equals the number of primary constraints. These parameters can be taken to be a function of time only. A method for solving these equations has been given in [5]. The final step consists in computing the variations (4), and in eliminating the canonical momenta and the Lagrange multipliers in terms of the coordinates and velocities using the first of the Hamilton equations of motion. In particular the multipliers are eliminated by making use of the Hamilton equations for the variables which are conjugate to the primary constraints. In fact Eq. (16) is just a consistency condition of the entire scheme, as we now show.

The primary constraints can always be expressed in the form [1]

$$\Phi_{a_1} = p_{a_1} - f_{a_1}(\{q^a\}, \{p_{a_2}\}), \quad (22)$$

where q^{a_1}, p_{a_1} are canonically conjugate pairs of variables, with $\{\dot{q}^{a_1}\}$ the (arbitrary) non projectible velocities. Taking the variation of

$$\frac{dq^{a_1}}{dt} \approx [q^{a_1}, H_c] + [q^{a_1}, \Phi_{b_1}] v^{b_1}, \quad (23)$$

we have

$$\begin{aligned} \delta \frac{dq^{a_1}}{dt} &\approx \delta [q^{a_1}, H_c] + [q^{a_1}, \Phi_{b_1}] \delta v^{b_1} \\ &+ \delta [q^{a_1}, \Phi_{b_1}] v^{b_1}. \end{aligned} \quad (24)$$

Using Eq. (9) we obtain for the l.h.s.

$$\begin{aligned} \delta \frac{dq^{a_1}}{dt} &= \frac{d}{dt} \delta q^{a_1} \approx \frac{d\epsilon^a}{dt} [q^{a_1}, \Phi_a] \\ &+ \epsilon^a ([q^{a_1}, \Phi_a], H_c) \\ &+ [q^{a_1}, \Phi_a], \Phi_{c_1} v^{c_1}. \end{aligned} \quad (25)$$

Making use of the Jacobi identity as well as of Eq. (17), one readily finds

$$\left(\delta v^{b_1} - \frac{d\epsilon^{b_1}}{dt} + \epsilon^a [V_a^{b_1} + v^{a_1} C_{a_1 a}^{b_1}] \right) [q^{a_1}, \Phi_{b_1}] \approx 0. \quad (26)$$

Recalling Eq. (22), and noting that the gauge transformations are defined only modulo the trivial ones, we make use of this freedom in order to obtain from (26) the strong relations Eq. (16).

To make contact with previous literature, we recall the conditions given in Refs. [10,11],⁵

$$[H_c, G] - \frac{\partial G}{\partial t} = h^{a_1} \Phi_{a_1}, \quad (27)$$

$$[\Phi_a, \Phi_{b_1}] = C_{ab_1}^{c_1} \Phi_{c_1}, \quad (28)$$

where G is of course always understood to be first-class. As was however emphasized in Ref. [5], the second condition is restrictive. Indeed, if we take Eq. (28) together with the Ansatz (5) as our starting point, we are led to

$$0 = \frac{d\epsilon^{b_2}}{dt} - \epsilon^a V_a^{b_2} \quad (29)$$

as the only condition. Note that this condition follows from our general relations (16), (17), since the structure functions $C_{a_1 a}^{b_2}$ are assumed to be zero, as implied by assumption (28). Note also that our first relation is absent since their conditions do not involve any Lagrange multipliers.

We now want to make contact with the purely Lagrangian methods of obtaining the gauge symmetries [1–4]. As discussed by Dirac [9], the classical Euler-Lagrange equations follow from the action principle $\delta S_T = 0$, where S_T is defined by

$$S_T = \int dt [p_i \dot{q}^i - H_T]. \quad (30)$$

We now show that the condition (13) does indeed ensure the invariance of the total action under the transformations generated by G . Consider $L_T = p_i \dot{q}^i - H_T$. Assuming the commutativity (9), we find for an infinitesimal transformation generated by G

$$\begin{aligned} \delta L_T &= [p_i, G] \dot{q}^i - \dot{p}_i [q^i, G] - [H_T, G] \\ &+ \frac{d}{dt} (p_i \delta q^i) \\ &= \frac{\partial}{\partial t} G + [G, H_T] + \frac{d}{dt} (-G + p_i \delta q^i). \end{aligned} \quad (31)$$

Since the endpoint configurations in the total action (30) are taken to be fixed, we see that the invariance of the total action under this transformation leads to the off-shell condition (13). Observe that no use has been made of the equations of motion.

The general variation of the Lagrangian $L(q, \dot{q})$ is given by

$$\delta L = -L_i \delta q^i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right), \quad (32)$$

where L_i is the Euler derivative, given in terms of the Hessian W_{ij} by

$$L_i = W_{ij} \ddot{q}^j + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \dot{q}^j - \frac{\partial L}{\partial q^i}. \quad (33)$$

Note that $-L_i \delta q^i$ in the present formulation corresponds to $\frac{\partial}{\partial t} G + [G, H_T] - \frac{d}{dt} G$ in the formulation in terms of the total action. Both expressions vanish on shell.

It is well known [1–4] that to each gauge symmetry of the Lagrangian there is a corresponding gauge identity having the general form

$$\Lambda_{a_1} = \sum_k \frac{d^k}{dt^k} (\rho_{(k)a_1}^i(q, \dot{q}) L_i) \equiv 0. \quad (34)$$

⁵ In Ref. [10], Eq. (28) was given in the form $[G, \Phi_{a_1}] = g_{a_1}^{b_1} \Phi_{b_1}$.

Taking a general gauge transformation of the form [2]⁶

$$\delta q^i = \sum_{k, a_1} (-1)^k \frac{d^k \eta^{a_1}(t)}{dt^k} \rho_{(k)a_1}^i(q, \dot{q}), \quad (35)$$

where $\eta^{a_1}(t)$ are the gauge parameters, one finds that the variation of the Lagrangian is given by

$$\delta L = -A_{a_1} \eta^{a_1} + \dots, \quad (36)$$

where the ‘‘dots’’ stand for a contribution given by a total time derivative, which does not contribute to the variation of the action. Because of the gauge identities the action is invariant. The corresponding statement in the case of the total action (30) is that, once Eqs. (17) are solved, the master equation (13) is satisfied identically without making use of the equations of motion. From the above point of view the difficulty in solving Eqs. (17) manifests itself in the problem of finding the zero modes of the Hessian, from which the functions $\rho_{(k)a_1}^i$ in Eq. (34) are determined.

The number of gauge identities is equal to the number of independent gauge parameters. This also coincides with the number of first class primary constraints. Thus, for Hamiltonian systems involving only such constraints, there are no restrictions on the parameters parametrizing the gauge generators (5). An important class of systems having this property are the so called ‘‘zero-Hamiltonian’’ systems where $H_c \equiv 0$, which characterizes reparametrization invariant theories. In that case the dynamics is described by $H_T = \sum \nu^{a_1} \Phi_{a_1}$, where the sum extends only over the primary first class constraints. The Dirac algorithm ensures that these are in fact the only first class constraints. Hence the gauge generator is described entirely in terms of these constraints. As a result we obtain only relations (16), and there are no restrictions on the infinitesimal gauge parameters.

For some physically interesting models, the structure functions V_b^a and $C_{a,b}^c$ are actually constants, and the gauge parameters are just functions of t . In the case where $C_{ab}^c = 0$ and the V_b^a 's are constant, Eqs. (17) have been solved in Ref. [12]. It leads to variations in the coordinates which coincide with the general form given in (35).

⁶ The form of δq^i as given by Eq. (35) is also suggested within the Hamiltonian framework by the work of Ref. [5], with η^{a_1} playing the role of the independent gauge parameters.

An example where the V_b^a depend on the coordinates is given by the pure Yang-Mills Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a. \quad (37)$$

This is a purely first class system with one primary constraint

$$\pi_0^a(x) \approx 0, \quad (38)$$

and one secondary constraint

$$[\mathcal{D}_i \pi_i]^a(x) = \partial_i \pi_i^a(x) + f_{abc} A_i^b(x) \pi_i^c(x) \approx 0. \quad (39)$$

The canonical Hamiltonian is given by

$$H_c = \int d^3x \left[\frac{1}{2} (\boldsymbol{\pi}^a)^2 + \frac{1}{4} (F_{ij}^a)^2 + A_0^a (D_i \pi_i)^a \right]. \quad (40)$$

From this expression one readily finds that the non-vanishing structure functions analogous to those in Eqs. (2) and (3) are given by⁷

$$\begin{aligned} (V_1^2(x, y))_{ab} &= \delta(x - y) \delta_{ab}, \\ (V_2^2(x, y))_{ab} &= g f_{acb} A^0 \delta(x - y), \\ (C_{22}^2(x, y, z))_{abc} &= -g f_{abc} \delta(x - y) \delta(y - z). \end{aligned} \quad (41)$$

Hence in the case of the pure Yang-Mills theory Eq. (17) reduces to

$$\partial_0 \epsilon_2^a(x) = \epsilon_1^a(x) + g f_{acb} A^{0b}(x) \epsilon_2^c(x). \quad (43)$$

Taking $\epsilon_2^a(x) = \alpha^a(x)$ as the independent gauge parameters and solving the above equation for ϵ_1^a leads to the following structure of the gauge generator:

$$G(x) = (\mathcal{D}_0 \alpha)^a(x) \pi_0^a(x) + (\mathcal{D}_i \pi_i)^a(x) \alpha^a(x). \quad (44)$$

From this we immediately obtain for the infinitesimal gauge transformations of the potentials the familiar result

$$\delta A_\mu^a(x) = \int d^3y [A_\mu^a(x), G(y)] = (\mathcal{D}_\mu \alpha)^a(x). \quad (45)$$

We can also compute the variation of the multiplier by using the first of the Hamilton equations to obtain

⁷ The index ‘‘1’’ and ‘‘2’’ refer to the primary and secondary constraint, respectively.

the relation $A_0^a = v^a$, and then using the commutativity analogous to Eq. (9). The result is

$$\delta v^a = \partial_0 \delta A_0^a = \partial_0 (\mathcal{D}_0 \alpha)^a. \quad (46)$$

The same equation also follows directly from Eq. (16) with the identification $\epsilon^{b_1} \rightarrow \epsilon_1^a(x) = [\mathcal{D}_0 \alpha(x)]^a$.

To complete our discussion we now reproduce the Lagrangian gauge identities following from our Hamiltonian analysis. It is easy to see that the variation (45) can be cast into the form of Eq. (35) with k taking the values $k = 0, 1$, where $\eta^{a_1}(t)$ are identified with the gauge parameters $\alpha^a(x)$, and

$$\begin{aligned} \rho_{(0)b}^{0a}(x, y) &= g_{acb}^f A^{0c} \delta^3(x - y), \\ \rho_{(1)b}^{0a}(x, y) &= -\delta_{ab} \delta^3(x - y), \\ \rho_{(0)b}^{ia}(x, y) &= \mathcal{D}_{ab}^i \delta^3(x - y). \end{aligned} \quad (47)$$

Using these expressions in Eq. (34), we arrive at

$$\Lambda_a(x) \equiv (\mathcal{D}^\mu L_\mu)_a(x) = 0 \quad (48)$$

where L_μ is the Euler derivative, which in the present case is given by $\mathcal{D}^\sigma F_{\sigma\mu}$. We have thus arrived at the standard gauge identity of the Yang-Mills Lagrangian.

To conclude, we wish to emphasize once more the main points of our paper. We have derived for the most general case a master equation in the Hamiltonian formalism, which expresses the time independence of the generator of gauge transformations, and compactly encodes a pair of equations giving the restrictions on the gauge parameters, as well as the variations of the Lagrange multipliers. We have further explicitly demonstrated the consistency of this pair of equations with Hamilton's equations of motion. The commutativity requirement (9) played a key role in the whole analysis. Observe that in a Lagrangian framework this commutativity is always used when deriving the equations of motion, or obtaining the gauge symmetries, while on the Hamiltonian level it implies non-trivial restrictions on the gauge parameters. The master equation was also shown to imply the invariance of the total action, as well as the covariance of the Hamilton equations of motion. We further discussed the connection with the purely Lagrangian approach. In

particular we established a correspondence between the gauge identities and the master equation, which vanishes identically when expressed in terms of the free parameters.

We have discussed here systems involving only first class constraints. The extension to mixed systems can be done in two distinct ways. (i) The conventional way would consist in replacing everywhere the Poisson brackets by the corresponding Dirac brackets. (ii) An alternative procedure would consist in embedding the original theory into a pure first class theory following the procedure of Ref. [13], and then to follow the steps given here. In particular, purely second class systems could also be treated in this way.

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