Quantum mechanical violation of macrorealism for large spin and its robustness against coarse-grained measurements

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For multilevel spin systems, robustness of the quantum mechanical (QM) violation of macrorealism (MR) with respect to coarse-grained measurements is investigated using three different necessary conditions of MR, namely, the Leggett-Garg inequality (LGI), Wigner's form of the Leggett-Garg inequality (WLGI), and the condition of no-signaling in time (NSIT). It is shown that for dichotomic sharp measurements, in the asymptotic limit of spin, the algebraic maxima of the QM violations of all these three necessary conditions of MR are attained. Importantly, the QM violations of all these persist in that limit even for *arbitrary* unsharp measurements, i.e., for any nonzero value of the sharpness parameter characterizing the degree of fuzziness of the relevant measurements. We also find that, when different measurement outcomes are clubbed into two groups for the sake of dichotomizing the outcomes, the asymmetry or symmetry in the number of outcomes in the two groups, signifying the degree of coarse graining of measurements, has a crucial role in discerning quantum violation of MR. The results clearly demonstrate that classicality does not emerge in the asymptotic limit of spin, whatever be the unsharpness and degree of coarse graining of the measurements.

I. INTRODUCTION

One of the central concepts underpinning the classical world view in the macroscopic domain is the notion of macrorealism (MR), which is characterized by the following two assumptions: *realism,* meaning that, at any instant, irrespective of measurement, a system is in any one of the available definite states such that all its observable properties have definite values; and *noninvasive measurability (NIM),* meaning that it is possible, in principle, to determine which of the states the system is in, without affecting the state itself or the system's subsequent evolution. That the conjunction of these two assumptions is in conflict with quantum mechanics was first shown by Leggett and Garg [1,2]. This was achieved by deriving from these assumptions a testable inequality involving time-separated correlation functions corresponding to successive measurement outcomes pertaining to a system whose state evolves in time. Such an inequality, known as the Leggett-Garg inequality (LGI), turns out to be incompatible with the relevant and testable quantum mechanical (QM) predictions. Thus, the LGI provides a necessary condition for MR, whose empirical violation would necessarily imply repudiation of MR. In recent years, investigations related to the LGI have been acquiring considerable significance, as evidenced by a wide range of theoretical and experimental studies; see, for example, a recent comprehensive review, Ref. [3].

Against the above backdrop, it is noteworthy that, apart from the LGI, of late, two more necessary conditions of MR have been proposed. One of them has been called Wigner's form of the LGI (WLGI) [4], and the other one is known as no-signaling in time (NSIT) [5]. WLGI can be regarded as a temporal version of Wigner's form of the local realist inequality [6,7] that is derived as a testable algebraic consequence of the probabilistic form of MR. On the other hand, the NSIT condition is formulated as a statistical version of NIM to be satisfied by any macrorealist theory.

In this paper, for any value of spin pertaining to *multilevel spin systems*, we focus on studying the robustness of the QM violations of the aforementioned necessary conditions of MR with respect to the measurement scheme introduced by Budroni and Emary [8] and considering fuzziness of the relevant measurements modeled in terms of what is known as "unsharp" measurement [9–13]. We have also adopted a more general approach to that measurement scheme by coarse graining different outcomes, well suitable for the purpose of studying quantum-classical transition in the macroscopic limit. Here we may note that, to date, there have been only a few studies on the QM violation of MR in the context of multilevel spin systems. One of these studies [14] shows that, for dichotomic measurements involving only two projectors that project a multilevel system onto one of the two possible subspaces, the maximum QM violation of the LGI for *any* dimensional system is the same as that for a qubit system. Another study [8] considers measurements involving projections onto individual levels and, interestingly, this study shows that the QM violation of the LGI increases with an increasing value of the spin of the system under consideration, with the algebraic maximum of the violation being attained for infinitely large value of spin. In the other study [15], how a suitable choice of the measurement scheme can lead to optimal violation of LGI for an arbitrary spin system has been shown, thus improving upon an earlier result [16]. Now, in the background of these studies, the key new results obtained in this paper are as follows.

(i) Similar to the case of the LGI, the QM violations of WLGI and NSIT for multilevel spin systems pertaining to measurements that involve projections onto individual levels, too, increase with an increasing value of the spin and the

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algebraic maxima of their violations are attained when the spin becomes infinitely large.

(ii) For modeling the effect of coarse graining of the measurements, when measurement outcomes of multilevel spin systems are dichotomized by clubbing different outcomes into two groups, say "+" and "−", then asymmetry or symmetry in the number of outcomes in the two groups has a crucial role in discerning quantum violation of MR. However, if the measurements are considered to be sharp, then these two groups of outcomes can be sharply distinguished, which is in general not true in the macroscopic limit.

(iii) We have, therefore, introduced unsharpness of the measurements to make the boundary between the two groups of outcomes imprecise. It is found that for arbitrarily large values of the spin, QM violations of all the three aforementioned necessary conditions of MR *persist* for *arbitrary* unsharpness of measurements associated with a lower degree of coarse graining of the relevant measurements. For extreme coarse-grained measurements, while there exists a threshold unsharpness of the measurements above which no QM violation of the LGI or WLGI occurs, the QM violation of NSIT persists for any amount of unsharpness of the measurements.

Significance of the abovementioned results stems from the following consideration. It may be recalled that among the various approaches suggested for addressing the issue of the classical limit of QM [17–29], there are two strands of prevalent wisdom that are relevant to the results obtained in this paper. One is that classical physics emerges from the predictions of QM in the so-called "macroscopic" limit either when the system under consideration is of high dimensionality, for example, a large spin system, or when a low-dimensional system is of large mass, or when it involves a large value of any other relevant parameter such as energy. The other is that classicality arises out of QM under the restriction of coarse-grained measurements for which one can empirically resolve only those eigenvalues of a relevant observable that are sufficiently well separated; in other words, this viewpoint stipulates that the limits of observability of quantum effects in an appropriate macroscopic limit determine the way the classicality emerges [16,30].

As regards the first approach mentioned above, we note that counterexamples questioning it have been pointed out. For instance, in the case of the Bell-EPR (Bell-Einstein-Podolosky-Rosen) scenario, it has been shown that quantum features in the sense of violating local realist inequalities persist in the "macroscopic" limit such as for the large number of constituents of the entangled system [31] or for the large dimensions of the constituents of the entangled system [32–34]. Further, for the Bell-EPR scenario, the QM violation of the relevant local realist inequalities seems to increase even in the limit of large numbers of particles and large magnitudes of spins considered together [35]. On the other hand, in the case of temporal correlations for which the violation of MR is probed through the violation of the LGI, all the relevant studies mentioned earlier [8,14,15] reveal that, irrespective of the nature of measurements, the QM violation of the LGI persists for arbitrary large values of spin of the system under consideration.

As regards the second approach mentioned above, it has been shown [16,30] that for a class of Hamiltonians governing the time evolution, if one goes into the limit of sufficiently large spins, but can experimentally only resolve eigenvalues which are separated by much more than the intrinsic quantum uncertainty, then the measurement outcomes appear to be consistent with those of classical laws. This is taken to suggest that classicality emerges out of QM under the restriction of coarse-grained measurements. Along this line of research there had been a number of investigations giving more insight into the nature of coarse graining of the measurements and the emergence of classicality or the persistence of quantumness. In Ref. [36] a micro-macro nonlocal correlation was established. Quantum violation of local realism has been shown [37] for entangled thermal states with very low detection efficiency, i.e., for the extreme coarse-grained measurement available. large amount of violation of the Bell inequality has been obtained [38] with the human eye as the detector in a micro-macro experiment and this violation is robust against photon loss. Precise (non-coarse-grained) measurements are shown [39] to be essential for demonstrating quantum features at the mesoscopic or macroscopic level and for observing nonlocality becoming more difficult with increasing system size. In Ref. [40], it has been discussed that quantum-classical transition is forced to occur when measurement references are coarsened, while this is not the case when the final projection is coarsened. This particular result has been discussed in detail at end of Sec. V. There are also other recent works in this direction [41].

Now, given the status of the abovementioned studies concerning the two approaches in question, there is an open question as to what happens if the question of emergence of classicality in the macroscopic limit is addressed for *arbitrary* values of the quantum number such as spin (including the asymptotic limit), *in conjunction* with taking into account the effect of coarse graining of the relevant measurements. We try to address this by examining the question of emergence of classicality in terms of the respective QM violations of all the three necessary conditions of MR, namely, the LGI, WLGI, and NSIT, modeling the effect of coarse graining of the relevant measurements and introducing the unsharpness of measurements characterized by a sharpness parameter. The striking result revealed by this study is that classicality does *not* emerge even in the asymptotic limit of spin by such coarse graining and unsharpness of measurements. Let us now broadly indicate the plan of this paper.

In the following section, we explain the relevant features of the system under consideration (an arbitrary spin system in a uniform magnetic field), the specific type of measurement scheme used, its generalization, and the way the fuzziness of the measurements is modeled by unsharp measurement. In Sec. III, the key results obtained using the LGI and WLGI are discussed, which is followed by Sec. IV pertaining to NSIT. In Sec. V, the key results obtained using the LGI, WLGI, and NSIT by generalizing the scheme by which different measurement outcomes are clubbed together into two different groups are discussed using projective measurement as well as using unsharp measurement. Finally, in the concluding section, Sec. VI, we elaborate a bit on the significance of the results obtained and also indicate directions for future studies.

II. SETTING UP OF THE MEASUREMENT CONTEXT

Consider a QM spin j system in a uniform magnetic field of magnitude, B_0, along the x direction. The relevant Hamiltonian is ($\hbar = 1$)

$$H = \Omega J_x, \tag{1}$$

where Ω is the angular precession frequency ($\propto B_0$) and J_x is the x component of the spin angular momentum. Consider measurements of the z component of spin (J_z) whose eigenvalues are denoted by m. The measurement scheme used here [8] has the following features.

(i) The quantity Q is such that $Q = -1$ when $m = -j$ and for any other value of m ranging from $-j+1$ to $+j$, $Q = +1$. We denote by Q_i and m_i the value of Q and the outcome of the J_z measurement, respectively, at instant t_i. Thus $Q_i = +$ (i.e., $Q_i = +1$) means $m_i = -j+1$ or $-j+2$ or, $\ldots$ $j-1$ or, j, and $Q_i = -$ (i.e., $Q_i = -1$) means $m_i = -j$. This grouping scheme of the measurement outcomes is used in Secs. III and IV.

(ii) We initialize the system so that at $t = 0$, the system is in the state $|-j; j\rangle$, where $|m; j\rangle$ denotes the eigenstate of the J_z operator with eigenvalue m.

(iii) Consider measurements of Q at times t_1, t_2, and t_3 ($t_1 < t_2 < t_3$) and set the measurement times as $\Omega t_1 = \Pi$ and $\Omega(t_2 - t_1) = \Omega(t_3 - t_2) = \frac{\Pi}{2}$. For any j, this choice of measurement times may not give the maximum quantum violation of the LGI, WLGI, or NSIT. However, this choice suffices to give an idea about the nature of QM violations of the relevant inequalities for large j.

(iv) We have also adopted a measurement scheme which is more general than described earlier and more natural in the context of emergence of classicality at the macroscopic limit with coarse-grained measurement. This is described as follows: $Q = -1$ for $m = -j, \ldots, -j+x$, and $Q = +1$ for $m = -j+x+1, \ldots, +j$, where $0 < x \leqslant$ integer part (j) and x is an integer.

The asymmetry in the number of measurement outcomes clubbed together decreases and, hence, the degree of coarse graining of the measurement increases with an increasing value of x. This generalized grouping scheme of the measurement outcomes is used in Sec. V. Here, for $x = 0$, the aforementioned scheme is reproduced, and $x =$ integer part (j) denotes the most macroscopic grouping scheme in the sense of describing the perfect coarse graining of the measurements.

Next, we use the notion of unsharp measurement in the context of treating fuzziness of the measurement. Unsharp measurement, a form of positive-operator-valued measurement (POVM), is well studied in the quantum formalism. In ideal sharp measurement, the probability of obtaining a particular outcome, say m in the case of J_z measurement, and the corresponding postmeasurement state are determined by the projector $P_m = |m; j\rangle\langle m; j|$. On the other hand, in the case of unsharp measurement, the probability of an outcome and the corresponding postmeasurement state are determined by the effect operator, which is defined as

$$F_m = \lambda P_m + (1-\lambda)\frac{\mathbb{I}}{d}, \tag{2}$$

where λ is the sharpness parameter, where $0 \leqslant \lambda \leqslant 1$, P_m is the projector onto the state $|m; j\rangle$, $\mathbb{I}$ is the identity operator, and d is the dimension of the system (for spin j system, $d = 2j+1$). Here $(1-\lambda)$ denotes the amount of white noise present in any unsharp measurement. Given the above specification of the effect operator, the probability of an outcome, say m, is given by $\mathrm{Tr}(\rho F_m)$ for which the postmeasurement state is given by $(\sqrt{F_m}\rho\sqrt{F_m}^{\dagger})/\mathrm{Tr}(\rho F_m)$, ρ being the state of the system on which measurement is done.

III. ANALYSIS USING LGI AND WLGI

For the purpose of this paper, we use the following form of a three-term LGI [3]:

$$K_{\mathrm{LGI}} = C_{12} + C_{23} - C_{13} \leqslant 1, \tag{3}$$

where $C_{ij} = \langle Q_i Q_j \rangle$ is the correlation function of the variable Q at two times, t_i and t_j.

As regards WLGI, since this has been introduced only recently [4], we briefly recapitulate its formulation before indicating the specific form of WLGI that we will be using in this paper. In the context of WLGI, the notion of realism implies the existence of overall joint probabilities $\rho(Q_1, Q_2, Q_3)$ pertaining to different combinations of definite values of observables or outcomes for the relevant measurements, while the assumption of NIM implies that the probabilities of such outcomes would be unaffected by measurements. Hence, by appropriate marginalization, the observable probabilities can be obtained. For example, the observable joint probability $P(Q_1-, Q_2+)$ of obtaining the outcomes -1 and $+1$ for the sequential measurements of Q at the instants t_1 and t_2, respectively, can be written as

$$\begin{aligned} P(Q_1-, Q_2+) &= \sum_{Q_3=\pm 1} \rho(-, +, Q_3) \\ &= \rho(-, +, +) + \rho(-, +, -). \end{aligned} \tag{4}$$

Writing similar expressions for the other measurable marginal joint probabilities $P(Q_1+, Q_3+)$ and $P(Q_2+, Q_3+)$, we get, for example, the following combination:

$$\begin{aligned} &P(Q_1+, Q_3+) + P(Q_1-, Q_2+) - P(Q_2+, Q_3+) \\ &\quad = \rho(+, -, +) + \rho(-, +, -). \end{aligned} \tag{5}$$

Then, invoking non-negativity of the joint probabilities occurring on the right-hand side of Eq. (5) the following form of WLGI is obtained in terms of three pairs of two-time joint probabilities:

$$\begin{aligned} K_{\mathrm{WLGI}} = &\, P(Q_2+, Q_3+) - P(Q_1-, Q_2+) \\ &- P(Q_1+, Q_3+) \leqslant 0. \end{aligned} \tag{6}$$

Similarly, other forms of WLGI involving any number of pairs of two-time joint probabilities can be derived by using various combinations of the observable joint probabilities. Here we consider the specific form of the three-term WLGI mentioned above [Eq. (6)].

For projective measurement. In order to calculate the expectation values and joint probabilities appearing in the aforementioned forms of LGI and WLGI, we proceed by writing the relevant time evolution operators as, for example, the time evolution operator from the initial time $t = 0$ to the instant of first measurement $t = t_1$, $U(t_1 - 0) = e^{-i\pi J_x} = R^2$

TABLE I. Table showing that the QM violations of the LGI and WLGI increase with increasing values of the spin for ideal sharp measurement.

j	$(K_{\mathrm{LGI}} - 1)$	$(K_{\mathrm{WLGI}} - 0)$
1	0.50	0.44
10	1.75	0.87
100	1.92	0.96

(where $R = e^{-i\frac{\pi}{2}J_x}$), and all the subsequent measurements are equispaced in time. Typically, any joint probability, for example, $P(Q_2+, Q_3+)$, for a spin j system is calculated using the Wigner D-matrix formalism and is of the form

$$P(Q_2+, Q_3+) = 1 - \frac{(4j)!}{4^{2j}[(2j)!]^2} + \frac{1}{2^{4j}} - \frac{1}{2^{2j}}. \quad (7)$$

Using such expressions, both K_{LGI} and K_{WLGI} can be evaluated. We then obtain in Eqs. (3) and (6), respectively,

$$K_{\mathrm{LGI}} = 3 + 4^{1-2j} - 4^{1-j} - \frac{2^{1-4j}(4j)!}{[(2j)!]^2}, \quad (8)$$

$$K_{\mathrm{WLGI}} = 1 + 4^{-2j} - 4^{-j} - \frac{4^{-2j}(4j)!}{[(2j)!]^2}. \quad (9)$$

QM violations of the LGI and WLGI are quantified by $(K_{\mathrm{LGI}} - 1)$ and $(K_{\mathrm{WLGI}} - 0)$, respectively. It is found that both these violations increase with increasing values of j. Specific results showing this feature for $j = 1, 10$, and 100 are given in Table I.

From Eqs. (8) and (9) it can be seen that for $j \to \infty$, $(K_{\mathrm{LGI}} - 1) \to 2$ [8] and $(K_{\mathrm{WLGI}} - 0) \to 1$. Thus, in both these cases, the algebraic maxima of both K_{LGI} and K_{WLGI} are attained for infinitely large spin value of the system under consideration.

For unsharp measurement. Next, considering in the context of unsharp measurement, the expression of a typical joint probability distribution is of the form given by,

$$P(Q_1+, Q_2-) = \sum_{k=-j+1}^{j} \mathrm{Tr}\big[F_{-j} U_{\Delta t_2} \sqrt{F_k} U_{\Delta t_1} \rho_i U^{\dagger}_{\Delta t_1} \sqrt{F_k}^{\dagger} U^{\dagger}_{\Delta t_2}\big], \quad (10)$$

where $\rho_i =$ initial state of the system $= |-j; j\rangle\langle -j; j|$, $U_{\Delta t_1} = U(t_1 - 0)$, and $U_{\Delta t_2} = U(t_2 - t_1)$.

Now, using the form of the effect operator defined earlier given by Eq. (2), the Hamiltonian mentioned in Eq. (1), and using the Wigner D-matrix formalism, one can obtain the joint probability pertaining to our measurement context as the following:

$$P(Q_1+, Q_2-) = \frac{x^2\lambda}{2^{2j}} + 2x\lambda\sqrt{\frac{1-\lambda}{2j+1}}\frac{1}{2^{2j}} + \frac{\lambda(1-\lambda)}{2j+1}\frac{2j}{2^{2j}} + \frac{x^2(1-\lambda)}{2j+1} + 2x\left(\frac{1-\lambda}{2j+1}\right)^{\frac{3}{2}} + \left(\frac{1-\lambda}{2j+1}\right)^2 2j, \quad (11)$$

where $x = \left(\sqrt{\frac{2j\lambda+1}{2j+1}} - \sqrt{\frac{1-\lambda}{2j+1}}\right)$. Using such joint probabilities, one can obtain

$$\begin{aligned} K_{\mathrm{LGI}} = {}& \frac{1}{(1+2j)^2[(2j)!]^2} 16^{-j}(\{16^j + 2(-2+16^j)\lambda^2 + 4j^2[16^j - 4^{1+j}\lambda + 2(2+16^j)\lambda^2] - 4\lambda(-2 + 4^j + 2\sqrt{1-\lambda} \\ & \times \sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} + 16^j\sqrt{1-\lambda}\sqrt{1+2j\lambda}) - 4j[16^j + 2\lambda(-2 + 2^{1+2j} - 2^{1+4j} \\ & + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} + 16^j\sqrt{1-\lambda}\sqrt{1+2j\lambda})]\} \\ & \times [(2j)!]^2 + 2(1+2j)\lambda(-2 + \lambda - 2j\lambda + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda})(4j)!) \end{aligned} \quad (12)$$

and

$$\begin{aligned} K_{\mathrm{WLGI}} = {}& \frac{1}{(1+2j)^2[(2j)!]^2} 16^{-j}(\{4j^2\lambda(-4^j + \lambda + 16^j\lambda) - \lambda(-2 + 4^j - 16^j + \lambda + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda} \\ & \times \sqrt{1+2j\lambda} + 2^{1+4j}\sqrt{1-\lambda}\sqrt{1+2j\lambda}) - 2j[16^j + \lambda(-2 + 2^{1+2j} - 3(16^j) \\ & + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} \\ & + 2^{1+4j}\sqrt{1-\lambda}\sqrt{1+2j\lambda})]\}[(2j)!]^2 + (1+2j)\lambda(-2 + \lambda - 2j\lambda + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda})(4j)!). \end{aligned} \quad (13)$$

Now, for a particular value of j, the ranges of λ for which the QM violations of the LGI and WLGI persist differ with the range for WLGI being greater than that for the LGI. Moreover, the robustness of QM violations of both the LGI and WLGI with respect to unsharpness of the measurement increases with increasing values of j. This is illustrated by the results given in Table II, which indicate that the ranges of λ for which the QM violations of the LGI and WLGI persist increase with increasing values of j.

Most interestingly, for $j \to \infty$, we get, for the QM violations of the LGI and WLGI, the following:

$$(K_{\mathrm{LGI}} - 1) \to 2\lambda^2 \quad (14)$$

and

$$(K_{\mathrm{WLGI}} - 0) \to \lambda^2, \quad (15)$$

which show that the ranges for which the QM violations of the LGI and WLGI persist become equal to (0,1]. On the

TABLE II. Table showing that the ranges of λ for which the QM violations of the LGI and WLGI persist for different spin values increase with increasing values of spin.

j	Ranges of λ for which the QM violation of LGI persists	Ranges of λ for which the QM violation of WLGI persists
1	(0.85,1]	(0.71,1]
10	(0.35,1]	(0.28,1]
100	(0.12,1]	(0.08,1]

other hand, for any j, magnitude of the QM violation of the LGI (WLGI) decreases for decreasing values of λ. This is illustrated by the results shown in Table III.

Thus it is shown that if one adopts the type of measurement scheme used here, in the macrolimit characterized by infinitely large spin values as well as for any nonzero value of the sharpness parameter (i.e., for an arbitrary degree of fuzziness of the relevant measurement), the QM violation of MR persists for both the LGI and WLGI.

For mixed initial state. The abovementioned results are obtained for the aforementioned pure initial state. Let us investigate whether such kind of behavior persists when the initial state becomes mixed, which is the more realistic situation involved in actually testing the macrolimit of quantum mechanics. Here, instead of taking the pure initial state $|-j;j\rangle$ at $t=0$, we initialize the system so that, at $t=0$, the system is in the state ρ given by

$$\rho = v|-j;j\rangle\langle -j;j| + (1-v)\frac{\mathbb{I}}{d}, \tag{16}$$

where v is the visibility parameter which changes the pure state into a mixed state and $(1-v)$ denotes the amount of white noise present in the state $|-j;j\rangle$ $(0 \leqslant v \leqslant 1)$, d is the dimension of the system, and $\frac{\mathbb{I}}{d}$ is the density matrix of the completely mixed state of dimension d. The minimum values of v for which QM violates different necessary conditions of MR signify the maximum amounts of white noise that can be present in the given state for the persistence of the QM violation of the relevant necessary condition of MR, and this value of v is known as the *threshold visibility* (v_{th}) pertaining to the given necessary condition of MR.

We take the Hamiltonian and choice of measurement times as described earlier and joint probabilities are evaluated for projective measurements. Using the Wigner D-matrix formalism, we obtain

$$K_{\text{LGI}} = \frac{1}{\{(1+2j)[(2j)!]^2\}}[16^{-j}(\{2^{(3+2j)} - (16^j)(3) + (2)[2-(3)2^{(1+2j)} + (3)(16^j)]v + 2j[16^j + 2(2-2^{(1+2j)} + 16^j)v]\}[(2j)!]^2 - 2(1+2j)v(4j)!)] \tag{17}$$

and

$$K_{\text{WLGI}} = \frac{1}{\{(1+2j)[(2j)!]^2\}}[16^{-j}(\{-4^j(-2+4^j) + [1+2^{(1+4j)} - (3)4^j + 2(1-4^j+16^j)j]v\} \times [(2j)!]^2 - (1+2j)v(4j)!)]. \tag{18}$$

For a particular value of j, the threshold visibility of the LGI and WLGI *differ* with the threshold visibility for WLGI being *smaller* than that for the LGI, which signifies that, for a particular j, QM violation of WLGI persists for a greater amount of mixedness compared to that of the LGI. Moreover, the threshold visibilities of both the LGI and WLGI *decrease* with *increasing* values of j. These results are shown in Table IV.

For any j, magnitudes of the QM violations of the LGI or WLGI become *smaller* for decreasing values of v, or increasing mixedness introduced in the initial state, while, for any fixed v (fixed amount of mixedness introduced in the initial state), magnitudes of the QM violations of the LGI or WLGI become *larger* for increasing values of j. These results are shown in Table V.

We also find for $j \to \infty$, the QM violations of the LGI and WLGI are given by

$$(K_{\text{LGI}} - 1) \to 2v \tag{19}$$

TABLE III. QM violations of the LGI and WLGI for different spin values j and different values of the sharpness parameter λ of the measurement.

	The magnitude of QM violation			
	of LGI for		of WLGI for	
j	$\lambda=0.7$	$\lambda=0.5$	$\lambda=0.7$	$\lambda=0.5$
10	0.59	0.19	0.31	0.12
50	0.80	0.37	0.40	0.19
100	0.85	0.41	0.43	0.21

TABLE IV. Table showing that the threshold visibilities of the LGI and WLGI decrease with increasing values of the spin for ideal sharp measurement.

j	v_{th} for LGI	v_{th} for WLGI
1	0.571	0.276
10	0.098	0.052
100	0.010	0.005

TABLE V. QM violations of the LGI and WLGI for different spin values and different mixedness incorporated in the initial pure state with ideal sharp measurement.

	Magnitude of the QM violation of					
	LGI for			WLGI for		
j	$v=0.8$	$v=0.6$	$v=0.4$	$v=0.8$	$v=0.6$	$v=0.4$
1	0.27	0.03	No violation	0.32	0.20	0.08
10	1.36	0.97	0.58	0.69	0.51	0.32
100	1.53	1.14	0.76	0.77	0.57	0.38

and

$$(K_{\mathrm{WLGI}} - 0) \to v. \tag{20}$$

These results clearly show that for very large j and for any amount of mixedness introduced in the initial state of the system, the QM violation of MR *persists* using the LGI or WLGI.

IV. ANALYSIS USING THE NSIT CONDITION

According to the NSIT condition, the measurement outcome statistics for any observable at any instant is independent of whether any prior measurement has been performed. In order to study the above condition, let us consider a system whose time evolution occurs between two possible states. Probability of obtaining the outcome -1 for the measurement of a dichotomic observable Q at an instant, say, t_3, without any earlier measurement being performed is denoted by $P(Q_3 = -1)$. NSIT requires that $P(Q_3 = -1)$ should remain unchanged even when an earlier measurement is made at t_2; i.e.,

$$P(Q_3 = -1) - [P(Q_2 = +1, Q_3 = -1) + P(Q_2 = -1, Q_3 = -1)] = 0. \tag{21}$$

QM violation of NSIT is quantified by the nonvanishing value of the left-hand side of Eq. (21).

For projective measurement. For the spin j system, for $H = \Omega J_x$, using the measurement scheme discussed in Sec. II and the choice of measurement times as well as of the initial condition mentioned there, we obtain, using the Wigner D-matrix formalism,

$$P(Q_3 = -1) - [P(Q_2 = +1, Q_3 = -1) + P(Q_2 = -1, Q_3 = -1)] = 1 - \frac{(4j)!}{4^{2j}[(2j)!]^2}. \tag{22}$$

It is found that the QM violation of NSIT increases with increasing values of j. This is illustrated by the representative results given in Table VI.

For $j \to \infty$, the QM violation of NSIT $\to 1$, which is the algebraic maximum of the left-hand side of the NSIT condition.

For unsharp measurement. For unsharp measurement defined in terms of the sharpness parameter λ for the spin j system described earlier, the left-hand side of Eq. (21) becomes

$$P(Q_3 = -1) - P(Q_2+, Q_3-) - P(Q_2-, Q_3-) = \frac{2^{-4j}\lambda}{(1+2j)[(2j)!]^2}[2 + (-1+2j)\lambda - 2\sqrt{1-\lambda}\sqrt{1+2j\lambda}]\{16^j[(2j)!]2 - (4j)!\}. \tag{23}$$

TABLE VI. Table showing that the QM violation of NSIT increases with increasing values of the spin for ideal sharp measurement.

j	Magnitude of the QM violation of NSIT
1	0.63
10	0.87
100	0.96

TABLE VII. QM violations of NSIT for different spin values j and different values of the sharpness parameter λ of the measurement.

	Magnitude of the QM violation of NSIT		
j	for $\lambda = 0.1$	for $\lambda = 0.5$	for $\lambda = 0.8$
1	0.0004	0.0521	0.2263
10	0.0026	0.1418	0.4502
100	0.0063	0.2085	0.5726
$\to \infty$	0.0100	0.2500	0.6400

It is then found that for an arbitrary value of spin j, the QM violation of NSIT persists for any nonzero value of the sharpness parameter λ.

For $j \to \infty$, for any λ, the left-hand side of the NSIT condition is given by

$$P(Q_3 = -1) - [P(Q_2 = +1, Q_3 = -1) + P(Q_2 = -1, Q_3 = -1)] \to \lambda^2. \tag{24}$$

Thus, even in the macrolimit characterized by $j \to \infty$, for any nonzero value of λ, the QM violation of MR persists using NSIT.

For a given λ, the QM violations of NSIT for different spin j systems differ, increasing with increasing values of j. Also, for a given j, magnitudes of the QM violations of NSIT increase with increasing values of λ, i.e., for increasing sharpness of the measurement. This is true even for infinitely large values of spin. These results are illustrated in Table VII.

For mixed initial state. Now, instead of taking the pure initial state $|-j; j\rangle$ at $t = 0$, we initialize the system so that, at $t = 0$, the system is in the state ρ given by Eq. (16). We take the Hamiltonian and choice of measurement times as described earlier and joint probabilities are evaluated for projective measurements. Using the Wigner D-matrix formalism, we obtain

$$K_{\mathrm{NSIT}} = v - \frac{2^{-4j} v (4j)!}{[(2j)!]^2}. \tag{25}$$

For any j, the magnitude of the QM violation of NSIT becomes *smaller* for decreasing values of v, or increasing mixedness introduced in the initial state, while, for any fixed v (fixed amount of mixedness introduced in the initial state), the magnitude of the QM violation of NSIT becomes *larger* for increasing values of j. These results are shown in Table VIII.

For any j (including arbitrarily large values of j), the threshold visibility of NSIT is 0. So, interestingly, for very large values of j and for any amount of mixedness introduced

TABLE VIII. QM violations of NSIT for different spin values and different mixedness incorporated in the initial pure state with ideal sharp measurement.

	Magnitude of the QM violation of NSIT		
j	for $v = 0.8$	for $v = 0.4$	for $v = 0.2$
1	0.50	0.25	0.13
10	0.70	0.35	0.17
100	0.77	0.38	0.19

in the initial state of the system, the QM violation of MR *persists* using the LGI, WLGI, or NSIT.

V. ANALYSIS USING LGI, WLGI, AND NSIT FOR A MORE GENERAL GROUPING SCHEME OF THE MEASUREMENT OUTCOMES

Here we generalize the scheme by which different measurement outcomes are clubbed together into two groups. In this case $Q = -1$ for $m = -j, \ldots, -j + x$, and $Q = +1$ for $m = -j + x + 1, \ldots, +j$, where $0 < x \leqslant$ integer part(j) and x is an integer. Here the degree of coarse graining of the measurement increases with increase in x. Any fixed value of x denotes a particular grouping scheme. We initialize the system so that at $t = 0$, the system is in the state $|-j; j\rangle$. We take the aforementioned Hamiltonian and choices of measurement times.

For projective measurement. Here joint probabilities appearing in the aforementioned particular form of the LGI, WLGI, or NSIT are calculated for ideal sharp measurement using the Wigner D-matrix formalism.

From numerical results it is found that for any j (also for arbitrarily large value), QM violation of the LGI exists for $x \leqslant$ integer part ($j - 1$) and no violation occurs for $x =$ integer part (j), whereas QM violations of WLGI and NSIT exist for any value of x, where $x \leqslant$ integer part (j). This indicates that QM violations of different necessary conditions of MR persist for very large degrees of coarse graining of the measurement. However, the magnitudes of the violations become *smaller* for increasing values of x, or increasing the degree of coarse graining of the measurement.

For a fixed and finite value of x, magnitudes of QM violations of the LGI, WLGI, or NSIT become *larger* for increasing values of j. For arbitrarily large values of j ($\frac{j}{x} \gg 1$), magnitudes of QM violations of the LGI, WLGI, or NSIT approach their respective algebraic maxima. However, the QM violations of different necessary conditions of MR approach their respective algebraic maxima *slowly* as one increases x. These results are shown in Table IX.

For unsharp measurement. Now, instead of projective measurement, let us employ unsharp measurement of the spin-z component observable. For this case the effect operators are defined as

$$F_m = \lambda P_m + (1 - \lambda)\frac{\mathbb{I}}{d}. \tag{26}$$

From numerical results it is observed that, for any j, the ranges of the sharpness parameter for which the QM violations of the LGI and WLGI persist become *smaller* for increasing values of x, or increasing the degree of coarse graining of the measurement. For a fixed and finite value of x, the ranges of the sharpness parameter for which the QM violations of the LGI and WLGI persist become *larger* for increasing values of j, and for arbitrarily large values of j ($\frac{j}{x} \gg 1$), both the ranges approach $(0, 1]$. However, these ranges approach $(0, 1]$ *slowly* as one increases x. This is shown in Table X.

TABLE IX. QM violations of the LGI, WLGI, and NSIT for different values of j and x with ideal sharp measurement.

	Magnitude of the QM violation of					
	LGI for		WLGI for		NSIT for	
j	$x = 10$	$x = 20$	$x = 10$	$x = 20$	$x = 10$	$x = 20$
40	1.52	1.32	0.76	0.66	0.76	0.66
60	1.61	1.46	0.81	0.73	0.81	0.72
80	1.67	1.53	0.83	0.77	0.83	0.76
100	1.70	1.58	0.85	0.79	0.85	0.79

TABLE X. Ranges of the sharpness parameter for which QM violations of the LGI, WLGI, and NSIT persist for different values of j and x.

	The range of sharpness parameter (λ) for which the QM violation of					
	LGI persists for			WLGI persists for		
j	$x = 5$	$x = 7$	$x = 9$	$x = 5$	$x = 7$	$x = 9$
10	(0.64, 1]	(0.75, 1]	(0.92, 1]	(0.53, 1]	(0.61, 1]	(0.72, 1]
20	(0.49, 1]	(0.55, 1]	(0.59, 1]	(0.40, 1]	(0.44, 1]	(0.48, 1]
30	(0.42, 1]	(0.47, 1]	(0.51, 1]	(0.33, 1]	(0.37, 1]	(0.40, 1]
40	(0.38, 1]	(0.42, 1]	(0.45, 1]	(0.29, 1]	(0.33, 1]	(0.36, 1]

Interestingly, the range of the sharpness parameter for which the QM violation of NSIT persists for arbitrary values of j (including arbitrarily large values of j) and arbitrary values of x is $(0, 1]$. This indicates that, surprisingly, for any particular scheme of branching of the outcomes (i.e., for a fixed value of x), for arbitrarily large values of j ($\frac{j}{x} \gg 1$), QM violations of all the necessary conditions of MR persist for almost any nonzero value of the sharpness parameter.

Mixed state of different total spins. Here it should be mentioned that in a macroscopic sample consisting of many microscopic spins, total spin of the ensemble may not be precisely defined in general. In these cases, instead of having total spin number j, the resulting state would be a mixture of different total spin numbers varying from 0 to j. For such a state using the measurement schemes discussed in Sec. II, the magnitude of QM violations of all the aforementioned necessary conditions of MR will be less than that for the state having well -defined total spin j. This is because for such a state, the magnitude of the QM violations will be the weighted average of the violations due to all possible total spins of the system varying from 0 to j. Now, from Tables I, VI, and IX it is clear that the magnitudes of the QM violations of the LGI, WLGI and NSIT decrease with decreasing values of spin. Hence, the QM violation of any of these necessary conditions of MR for such a mixed state will have to be less than that for a system having a precise total spin.

Coarsening of the measurement times. Here it should be noted that in this work we have not considered the coarse graining of measurement times. For the sake of completeness, it needs to be mentioned that in Ref. [40] the effect of coarsening of the measurement times has been discussed in detail. It has been shown that when coarse graining at the level of measurement outcome fails to reproduce classical behavior, coarsening of the measurement time, which is a particular example of measurement "reference" [40], can reproduce it. In the context of temporal correlations, coarsening of measurement time means unsharpness of measurement time over a range whose effect is taken into account by considering measurement of a suitably averaged observable. For this kind of measurement, depending upon the Hamiltonian, there exists a threshold value of the unsharpness of measurement time Δ_{th} above which the QM violation of MR in terms of the LGI disappears for any value of spin.

VI. CONCLUDING DISCUSSION

The grouping scheme of the measurement outcomes used in the first part of this paper was invoked by Budroni and Emary [8] to show that the magnitude of the QM violation of the LGI increases with increasing values of the spin and approaches the algebraic maxima in the limit of arbitrarily large spin values. In this paper, contingent upon using this grouping scheme, it is shown that the aforementioned result holds for the WLGI and NSIT conditions of MR as well, and importantly, even in the context of unsharp measurement, the QM violation of MR persists in the arbitrarily large value of the spin for an arbitrary sharpness of the relevant measurement. These results hold good even when we generalize the grouping scheme. Here the clubbing of the measurement outcomes into two groups makes the measurement coarse grained. However, the boundary between the two groups of outcomes remains precise, which is, in general, not true in the realization of the macrolimit. Employing, in conjunction, unsharp measurement makes this boundary also imprecise. Thus, simultaneously clubbing different measurement outcomes together and invoking unsharp measurement enables us to describe in a more natural way the coarse graining of the measurements. It is, therefore, emphatically demonstrated that classicality does not emerge for such coarse-grained measurement even for arbitrarily large spin values of the system.

An interesting upshot of the results obtained in this paper is that, for any particular grouping scheme of the measurement outcomes, the range of the sharpness parameter for which the QM violation of WLGI persists is greater than that of the LGI. This indicates that given a spin value, there is a range of the sharpness parameter for which the QM violation of MR can be tested using WLGI, but not in terms of the LGI. Interestingly, the ranges of the sharpness parameter for which the QM violations of WLGI and the LGI persist increase with increasing values of the spin. On the other hand, the QM violation of NSIT persists for any nonzero value of the sharpness parameter for any arbitrary spin value.

Finally, we recall that in the investigations of this paper we have used the model of the unsharp measurement that defines the effect operator in terms of a single parameter, i.e., the sharpness parameter [9–13]. A possible alternative way to model the unsharp measurement is to define the effect operator in terms of what is called the "biasness parameter" [42,43] along with the sharpness parameter. It should be instructive to investigate to what extent the results obtained in this paper would be affected by using such a two-parameter model of the unsharp measurement. Another line of future investigation would be to explore the implications of coarse graining of the measurement times mentioned in Ref. [40] in the context of the work presented in this paper.

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