Spin-dependent observable effect for free particles using the arrival time distribution

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The mean arrival time of free particles is computed using the quantum-mechanical probability current. This is uniquely determined in the nonrelativistic limit of Dirac equation, although the Schrödinger probability current has an inherent nonuniqueness. Since the Dirac probability current contains a spin-dependent term, an arrival time distribution based on the probability current shows an observable spin-dependent effect, even for free particles. This arises essentially from relativistic quantum dynamics, but persists even in the nonrelativistic regime.

I. INTRODUCTION

The treatment of time in quantum mechanics is a much debated question. A testimony to this is the proliferation of recent papers [1] on the problems of tunneling time, decay time, dwell time, and the arrival time. In this paper we are specifically concerned with the issue of arrival time [2].

In classical mechanics, a particle follows a definite trajectory; hence the time at which a particle reaches a given location is a well-defined concept. On the other hand, in standard quantum mechanics, the meaning of arrival time is rather problematic. Indeed, there exists an extensive literature on the treatment of arrival time distribution in quantum mechanics [3].

Using the Born interpretation, $|\psi(\mathbf{x},t_1)|^2$, $|\psi(\mathbf{x},t_2)|^2 \ldots$ give the position probability distributions at different instants t_1, $t_2 \ldots$. Now, the question that immediately arises is that if we fix the positions at $\mathbf{x}=\mathbf{X}_1$, $\mathbf{X}_2 \ldots$, can the functions $|\psi(\mathbf{X}_1,t)|^2$, $|\psi(\mathbf{X}_2,t)|^2 \ldots$ give the time probability distributions at different positions $\mathbf{X}_1$, $\mathbf{X}_2 \ldots$? It is well known that if at any instant $t=t_i$, $\int_{-\infty}^{+\infty}|\psi(\mathbf{x},t=t_i)|^2 d^3x=1$, the probability of finding the particle anywhere at that instant is unity. But if we fix the position at, say, $\mathbf{x}=\mathbf{X}_1$ and t is varied, the value of the integral $\int_0^{\infty}|\psi(\mathbf{x}=\mathbf{X}_1,t)|^2 dt \neq 1$. In this case what may be pictured is that at a given point, say, $\mathbf{X}_1$ the relevant probability changes with time and this change of probability is governed by the following continuity equation which suggests a "flow of probability":

$$\frac{\partial}{\partial t}|\psi(\mathbf{x},t)|^2+\nabla\cdot\mathbf{J}(\mathbf{x},t)=0, \tag{1}$$

where $\mathbf{J}(\mathbf{x},t)=(i\hbar/2m)(\psi\nabla\psi^*-\psi^*\nabla\psi)$ is the probability current density.

Different approaches for analyzing the problem of arrival time distribution have been suggested using the path integrals and positive-operator-valued measures [4]. A straightforward procedure would be to try to construct a self-adjoint operator for the arrival time in quantum mechanics which is conjugate to the Hamiltonian, but then it has been shown that such an operator does not have a basis of orthonormal eigenstates [5]. However, Delgado and Muga [6] have proposed an interesting approach by constructing a self-adjoint operator having dimensions of time which is relevant to the arrival time distribution, but then its conjugate Hamiltonian has an unbounded spectrum. The implications of this approach have been studied in detail by Delgado [7].

In this paper we adopt the definition of arrival time distribution in terms of the quantum probability current density $\mathbf{J}(\mathbf{x}=\mathbf{X},t)$. Interpreting the equation of continuity in terms of the flow of physical probability, the Born interpretation for the squared modulus of the wave function and its time derivative suggest that the mean arrival time of the particles reaching a detector located at $\mathbf{X}$ may be written as

$$\overline{\tau}=\frac{\int_0^{\infty}|\mathbf{J}(\mathbf{x}=\mathbf{X},t)|t\,dt}{\int_0^{\infty}|\mathbf{J}(\mathbf{x}=\mathbf{X},t)|dt}. \tag{2}$$

However, we emphasize that the definition of the mean arrival time used in Eq. (2) is *not* a uniquely derivable result within standard quantum mechanics. It should also be noted that $\mathbf{J}(\mathbf{x},t)$ can be negative, hence one needs to take the modulus sign in order to use the above definition. However, the Bohmian model of quantum mechanics in terms of the causal trajectories of individual particles implies the above expression for the mean arrival time in a unique way [8].

Although the quantum probability current interpreted as the streamlines of a conserved flux has been used for studying the tunneling times of Dirac electrons [9], it is easily seen that in nonrelativistic quantum mechanics the form of the probability current density is *not unique*, a point which has been explored by a number of authors [10–12]. If we replace $\mathbf{J}$ by $\mathbf{J}'$ in Eq. (1) where $\mathbf{J}'=\mathbf{J}+\delta\mathbf{J}$, with $\nabla\cdot\delta\mathbf{J}=0$, $\mathbf{J}'$ satisfies the same probability conservation as given by Eq. (1). Then this new current density $\mathbf{J}'$ will lead to a different distribution function for the arrival time [12]. Hence the question arises *how* one can uniquely fix the ar-

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rival time distribution via the quantum probability current in the regime of nonrelativistic quantum mechanics?

In order to address the above question, we take a vital clue from the interesting result that Holland [13] showed in the context of analyzing the uniqueness of the Bohmian model of quantum mechanics, viz., that the Dirac equation implies a *unique* expression for the probability current density for spin-$\frac{1}{2}$ particles in the nonrelativistic regime. In Sec. II we highlight the feature that the uniqueness of the probability current density is a *generic* consequence of *any* relativistic equation of quantum dynamics. In Sec. III, the particular case of the spin-dependent probability current density as derived from the Dirac equation is discussed. Subsequently, in Sec. IV, using the nonrelativistic limit of the Dirac current density, we compute the effect of spin on the arrival time distribution of free particles for an initial Gaussian wave packet. Such a line of investigation has not been explored sufficiently; to the best of our knowledge, only Leavens [14] has studied this issue specifically in terms of the Bohmian causal model of spin-$\frac{1}{2}$ particles.

II. UNIQUENESS OF THE PROBABILITY CURRENT DENSITY FOR ANY RELATIVISTIC WAVE EQUATION

The probability current obtained from any consistent relativistic quantum wave equation has to satisfy the *covariant* form of the continuity equation $\partial_\mu j^\mu = 0$, where the zeroth component of j^μ is associated with the probability density. Now, let us replace j^μ by $\overline{j^\mu}$ which should again be conserved, i.e., $\partial_\mu \overline{j^\mu} = 0$, where $\overline{j^\mu} = j^\mu + a^\mu$, a^μ being an arbitrary four-vector. But then the zeroth component $\overline{j^0}$ will have to reproduce the same probability density j^0, and hence $a^0 = 0$. This current as seen from another Lorentz frame is $j^{\mu\prime} = j^\mu + a^{\mu\prime}$. Then in this frame $j^{0\prime} = j^0 + a^{0\prime}$, and again from the previous argument $a^{0\prime} = 0$. But we know that the only four-vector whose fourth component vanishes in all frames is the null vector. Hence $a^\mu = 0$.

Thus, for any consistent relativistic quantum wave equation satisfying the covariant form of the continuity equation, the relativistic current is uniquely fixed. Unique expressions for the conserved currents have been explicitly derived by Holland [15] for the Dirac equation, the Klein-Gordon equation, and also for the coupled Maxwell-Dirac equations.

Now, an interesting point is that this uniqueness will be preserved in the nonrelativistic regime. Hence, given *any* relativistic wave equation, one can calculate the unique form of the current which can be used in the nonrelativistic regime. Then using the (normalized) modulus of the probability current density as the arrival time distribution, if one calculates the mean arrival time, it can be used to empirically test *any* relativistic wave equation, such as the relativistic Kemmer equation [16] for the massive spin-0 and spin-1 bosons.

Of late, a unique form of the probability current density expression has been derived in the nonrelativistic limit of the relativistic Kemmer equation for spin-0 and spin-1 particles [17]. Although the general scheme we outline for testing a relativistic quantum wave equation in terms of the arrival time distribution is not contingent on any specific form of the relativistic wave equation, in the following detailed study we specifically use the Dirac equation for spin-$\frac{1}{2}$ particles.

III. SPIN-DEPENDENT EFFECT ON THE ARRIVAL TIME DISTRIBUTION USING DIRAC EQUATION

The Dirac equation for a *free particle* is

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{\hbar c}{i} \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2 \right) \psi, \tag{3}$$

where

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

ψ is a four-component column matrix and σ_i are the Pauli matrices. Choosing a representation where ψ_1 and ψ_2 are two-component spinors, one gets two coupled equations

$$\frac{\partial \psi_1}{\partial t} = -c\sigma_i \frac{\partial \psi_2}{\partial x^i} - \frac{i m_0 c^2}{\hbar} \psi_1, \tag{4}$$

$$\frac{\partial \psi_2}{\partial t} = -c\sigma_i \frac{\partial \psi_1}{\partial x^i} + \frac{i m_0 c^2}{\hbar} \psi_2. \tag{5}$$

Combining Eqs. (4) and (5) one gets

$$\frac{\partial}{\partial t}(\psi_1^\dagger \psi_1) = -c\psi_1^\dagger \sigma_i \frac{\partial \psi_2}{\partial x^i} - c\frac{\partial \psi_2^\dagger}{\partial x^i} \sigma_i \psi_1. \tag{6}$$

For positive energies, one can take $\psi_2 \propto \exp(-iEt/\hbar)$, where E is the total energy. In the nonrelativistic limit, $E \cong m_0 c^2$ and then we have $E + m_0 c^2 \cong 2 m_0 c^2$. Then using this with Eq. (5) one can write

$$\psi_2 = -\frac{i\hbar c}{(E + m_0 c^2)} \sigma_i \frac{\partial \psi_1}{\partial x^i} = -\frac{i\hbar}{2 m_0 c} \sigma_i \frac{\partial \psi_1}{\partial x^i}. \tag{7}$$

Putting this value of ψ_2 in Eq. (6), one gets

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{8}$$

where $\mathbf{J}$ is the Dirac current in the nonrelativistic limit that can be decomposed into two terms, as was shown by Holland [13,15], as

$$\begin{aligned} \mathbf{J} &= -\frac{i\hbar}{2m}[\psi_1^\dagger \sigma(\sigma \cdot \nabla)\psi_1 - (\nabla \psi_1^\dagger \cdot \sigma)\sigma \psi_1] \\ &= -\frac{i\hbar}{2m}[\psi_1^\dagger(\nabla \psi_1) - (\nabla \psi_1^\dagger)\psi_1] + \frac{\hbar}{2m}\nabla \times (\psi_1^\dagger \sigma \psi_1) \end{aligned} \tag{9}$$

and $\rho = \psi_1^\dagger \psi_1$. ψ_1 is a two-component spinor which can be written for a particle in a spin eigenstate as

$$\psi_1 = \psi(x,t)\chi \equiv \left[R(x,t)\exp\left(\frac{iS(x,t)}{\hbar}\right)\right]\chi. \qquad (10)$$

Here $\psi(x,t)$ is the Schrödinger wave function and χ is a spin eigenstate. Putting this form of ψ_1 in the expression for current in Eq. (9) one gets

$$\mathbf{J} = \frac{1}{m}\rho \nabla S + \frac{1}{m}(\nabla \rho \times \mathbf{s}) \equiv \mathbf{J}_i + \mathbf{J}_s \qquad (11)$$

with

$$\mathbf{s} = (\hbar/2)\chi^\dagger \sigma \chi, \quad \rho = R^2, \quad \chi^\dagger \chi = 1.$$

The first term $\mathbf{J}_i$ in Eq. (11) is independent of spin, while the second term $\mathbf{J}_s$ contains the contribution of the spin of a free particle to the unique conserved vector current in the nonrelativistic limit. Now, since the mean arrival time given by Eq. (2) can be computed by using the unique expression for $\mathbf{J}$ in Eq. (11), one can thus obtain a spin-dependent contribution to the expression for the mean time of arrival for free particles. This could be experimentally measurable. On the other hand, if one ignores the spin-dependent term one would obtain the mean arrival time given by

$$\overline{\tau_i} = \frac{\int_0^\infty |\mathbf{J}_i| t\, dt}{\int_0^\infty |\mathbf{J}_i| dt}. \qquad (12)$$

In the following section we will study the situations where the difference between the magnitudes of $\overline{\tau}$ and $\overline{\tau_i}$ is significant, thereby enhancing the feasibility of detecting the predicted spin-dependent effect.

IV. THE COMPUTED EFFECTS ON THE ARRIVAL TIME DISTRIBUTION

We consider a freely evolving Gaussian wave packet in the two separate cases (A and B) corresponding to an initially *symmetric* and an *asymmetric* wave packets, respectively.

1. Case A: Symmetric wave packet

Let us consider a Gaussian wave packet for a free spin-$\frac{1}{2}$ particle of mass m centered at the point $x=0$, $y=0$, and $z=0$. We choose the spin to be directed along the z axis, i.e. $(\mathbf{s}=\frac{1}{2}\hat{z})$:

$$\psi(\mathbf{x},t=0) = \frac{1}{(2\pi\sigma_0^2)^{3/4}} \exp(i\mathbf{k}\cdot\mathbf{x}) \exp\left(-\frac{\mathbf{x}^2}{4\sigma_0^2}\right). \qquad (13)$$

The time evolved wave function can be written as

$$\psi(\mathbf{x},t) = R(\mathbf{x},t)\exp\left[\frac{iS(\mathbf{x},t)}{\hbar}\right], \qquad (14)$$

where

$$R(\mathbf{x},t) = (2\pi\sigma^2)^{-3/4}\exp\left[-\frac{(\mathbf{x}-\mathbf{u}t)^2}{4\sigma^2}\right] \qquad (15)$$

and

$$S(\mathbf{x},t) = -\frac{3\hbar}{2}\tan^{-1}\left(\frac{\hbar t}{2m\sigma_0^2}\right) + m\mathbf{u}\cdot\left(\mathbf{x} - \frac{1}{2}\mathbf{u}t\right) + \frac{(\mathbf{x}-\mathbf{u}t)^2\hbar^2 t}{8m\sigma_0^2\sigma^2} \qquad (16)$$

with $\mathbf{u}=\hbar\mathbf{k}/m$, the initial group velocity taken along the x axis, and

$$\sigma = \sigma_0\left[1 + \frac{\hbar^2 t^2}{4m^2\sigma_0^4}\right]^{1/2}. \qquad (17)$$

The total current density can be calculated using Eq. (11) to be (we set $m=1$, $\hbar=1$)

$$\mathbf{J} = \rho\left[\left(u + \frac{(x-ut)t}{4\sigma_0^2\sigma^2}\right)\hat{\mathbf{x}} + \left(\frac{yt}{4\sigma_0^2\sigma^2}\right)\hat{\mathbf{y}} + \left(\frac{zt}{4\sigma_0^2\sigma^2}\right)\hat{\mathbf{z}}\right] + \rho\left[-\left(\frac{y}{2\sigma^2}\right)\hat{\mathbf{x}} + \frac{(x-ut)}{2\sigma^2}\hat{\mathbf{y}}\right], \qquad (18)$$

where the contribution of spin is contained in the second term only.

We can now compute $\overline{\boldsymbol{\tau}}$ and $\overline{\boldsymbol{\tau}_i}$ numerically by substituting Eq. (18) in Eqs. (2) and (12), respectively. It is instructive to examine the behavior of the contribution of spin-dependent term towards the mean arrival time. For this purpose, we define a quantity

$$\overline{\boldsymbol{\tau}_s} = \frac{\int_0^\infty |\mathbf{J}_s| t\, dt}{\int_0^\infty |\mathbf{J}_s| dt}. \qquad (19)$$

We first compute $\overline{\boldsymbol{\tau}_s}$ for a range of the initial velocity u in units of $m=1$, and $\hbar=1$. We find that the spin of a free particle contributes towards altering its mean arrival time for a wide range of initial velocities. This feature holds generally, except for very small magnitudes of velocity where the spin-dependent contribution may be negligible depending on the location of the detector vis-a-vis the direction of the initial group velocity $\mathbf{u}$. This feature is shown in Fig. 1 where we plot the variation of $\overline{\tau_s}$ with u. The initial wave packet is peaked at the origin with $\sigma_0=0.01$. The detector position is chosen at $(x=1,\ y=1,\ z=1)$. We find that the difference of magnitude between $\overline{\tau}$ and $\overline{\tau_i}$ can be increased by choosing asymmetric detector positions as well as asymmetric spread for the initial wave packet, an example of which we will now discuss.

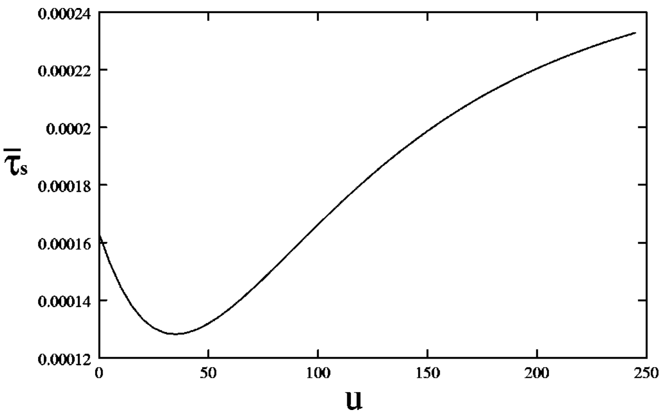

FIG. 1. The spin-dependent contribution to the mean arrival time computed at the point $x=1,\ y=1,\ z=1$ is plotted against the initial group velocity of the packet along the x axis (in units of $\hbar=1=m$).

2. Case B: Asymmetric wave packet

We consider an initial free particle wave packet in three dimensions which is centered at the point $x=-x_1,\ y=0,\ z=0$:

$$\psi(x,y,z,t=0)=\left(\frac{1}{\pi^3 a^2 b^2 c^2}\right)^{1/4}\exp(ikx)\exp\left[-\frac{(x+x_1)^2}{2a^2}\right]\times\exp\left[-\frac{(y)^2}{2b^2}\right]\exp\left[-\frac{(z)^2}{2c^2}\right], \tag{20}$$

where a,b,c are positive constants. (Such a form for the wave function was considered by Finkelstein [12] in the context of arrival time distributions.) The particle is given an initial velocity in the x direction represented by $u=\hbar k/m$. The time evolved wave function is given by

$$\psi(x,y,z,t)=\left(\frac{a^2b^2c^2}{\pi^3}\right)^{1/4}\frac{\exp[i(kx-k^2t/2)]}{\alpha\beta\gamma}\times\exp\left[-\frac{(x+x_1-kt)^2}{2\alpha^2}\right]\exp\left[-\frac{y^2}{2\beta^2}\right]\times\exp\left[-\frac{z^2}{2\gamma^2}\right], \tag{21}$$

where $\alpha=(a^2+it)^{1/2},\ \beta=(b^2+it)^{1/2},\ \gamma=(c^2+it)^{1/2}$.

Writing the wave function as

$$\psi(x,y,z,t)=R(x,y,z,t)\exp\left[\frac{iS(x,y,z,t)}{\hbar}\right] \tag{22}$$

one obtains

$$R(x,y,z,t)=\left(\frac{a^2b^2c^2}{\pi^3}\right)^{1/4}\frac{1}{(p^2+q^2)^{1/4}}\times\exp\left[-\frac{a^2(x+x_1-kt)^2}{2(a^4+t^2)}\right]\times\exp\left[-\frac{b^2y^2}{2(b^4+t^2)}\right]\exp\left[-\frac{c^2z^2}{2(c^4+t^2)}\right] \tag{23}$$

and

$$S(x,y,z,t)=\hbar kx-\frac{\hbar k^2 t}{2}-\frac{\hbar}{2}\tan^{-1}(q/p)+\frac{\hbar t(x+x_1-kt)^2}{2(a^4+t^2)}+\frac{\hbar t y^2}{2(b^4+t^2)}+\frac{\hbar t z^2}{2(c^4+t^2)} \tag{24}$$

with

$$p=a^2b^2c^2-a^2t^2-b^2t^2-c^2t^2,$$
$$q=a^2b^2t+a^2c^2t+b^2c^2t-t^3. \tag{25}$$

Considering again a spin-$\frac{1}{2}$ particle with spin directed along the z axis ($\mathbf{s}=\frac{1}{2}\hat{z}$), the total current density defined in Eq. (11) is given by (in units of $\hbar=1=m$)

$$\mathbf{J}=\rho\left[\left(u+\frac{(x+x_1-ut)t}{(a^4+t^2)}\right)\hat{\mathbf{x}}+\frac{yt}{(b^4+t^2)}\hat{\mathbf{y}}+\frac{zt}{(c^4+t^2)}\hat{\mathbf{z}}\right]+\rho\left[-\frac{b^2y}{(b^4+t^2)}\hat{\mathbf{x}}+\frac{a^2(x+x_1-ut)}{(a^4+t^2)}\hat{\mathbf{y}}\right], \tag{26}$$

where the second term represents the spin-dependent contribution to the current.

We compute numerically the arrival times $\overline{\tau}$ and $\overline{\tau_i}$. Figure 2 shows the variation of $\overline{\tau}$ and $\overline{\tau_i}$ with the initial group velocities u of the wave packet. Here we choose the parameters as $x_1=0,\ a=0.001,\ b=0.4,\ c=0.01$. Accordingly the mean arrival time is computed at the position $x=1.0,\ y=2.0,\ z=1.0$. One sees that the difference in the magnitudes of $\overline{\tau}$ and $\overline{\tau_i}$ can be suitably enhanced by a judicious choice of asymmetric initial spreads and detector positions.

V. CONCLUDING REMARKS

Let us now summarize the salient features of our scheme. For measuring the spin of a particle, it is usually subjected to an external field, as in a Stern-Gerlach apparatus. But the scheme we suggest would enable to detect a spin-dependent effect *without* using any external field. Such an observable effect thus highlights the feature that the spin of a particle is an *intrinsic* property and is *not* contingent on the presence of an external field. As demonstrated in this paper, the spin-dependent term in the Dirac probability current density *con-*

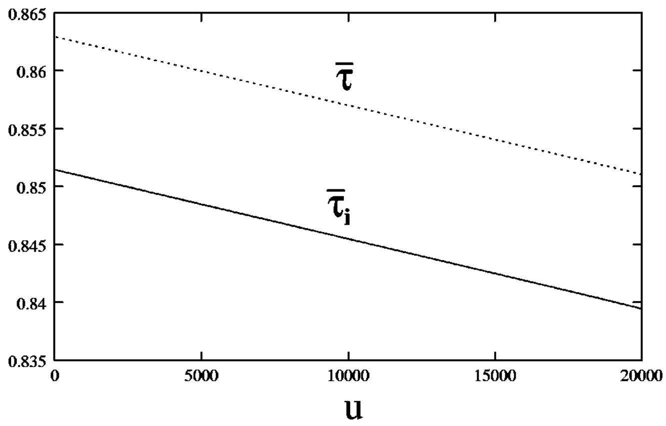

FIG. 2. The mean arrival times $\overline{\tau}$ (upper curve) and $\overline{\tau_i}$ (lower curve) computed at the point $x=1$, $y=2$, $z=1$ are plotted against the initial group velocity of the packet along the x axis. and $a=0.001$, $b=0.4$, $c=0.01$, $x_1=0$ (in units of $\hbar=1=m$).

tributes significantly to the computed mean arrival time for a range of suitably chosen parameters of the Gaussian wave packet. Thus if the arrival time distribution can be measured, this predicted spin-dependent effect would be empirically verifiable.

One may also perceive the significance of such an effect as follows. Although the dynamical properties of free particles such as position, momentum, and energy are measurable, one cannot measure the *static* or *innate* particle properties such as charge without using any external field. Nevertheless, the scheme we have discussed shows that the magnitude of total spin can be measured *without* subjecting the particle to an external field.

Another implication of measuring the spin-dependent arrival times for free particles could be to view this as implying an interesting difference between the magnitudes of the total spin of a particle and its other static properties such as mass and charge. This is because the measurability of the property of spin of a free particle arises from the relativistic nature of the dynamical evolution of the wave function where the relevant wave function is fundamentally four-component (or two-component), even in the *nonrelativistic limit*.

Now, since the spin-dependent term which contributes significantly to the arrival time distribution has been computed in the nonrelativistic regime by starting from the relativistic Dirac equation, this provides a rather rare example of an empirically detectable manifestation of a relativistic dynamical equation in the *nonrelativistic regime*. This effect *cannot* be derived *uniquely* from the Schrödinger dynamics.

A future line of investigation as an offshoot of this paper could be to explore the possibilities of using the relativistic quantum-mechanical wave equations of particles with spins other than spin $\frac{1}{2}$ (such as using the Kemmer equation [16,17] for spin-0 and spin-1 bosons) in order to compute the spin-dependent terms in the probability current densities and their effects on the arrival time distribution. Such a study seems worthwhile because the arrival time distribution may provide a means of checking the validity of the various suggested relativistic quantum-mechanical equations which have otherwise eluded any empirical verification.

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[1] See, for example, relevent recent reviews in *Time in Quantum Mechanics*, edited by J.G. Muga, R. Sala Mayato, and I.L. Egusquiza (Springer-Verlag, Berlin, 2002).

[2] For an in-depth review, see, J.G. Muga and C.R. Leavens, Phys. Rep. **338**, 353 (2000).

[3] C.R. Leavens, Phys. Lett. A **178**, 27 (1993); J.G. Muga, S. Brouard, and D. Macias, Ann. Phys. (N.Y.) **240**, 351 (1995); Y. Aharanov, J. Oppenheim, S. Popescu, B. Reznik, and W.G. Unruh, Phys. Rev. A **57**, 4130 (1998); J.J. Halliwell and E. Zafiris, Phys. Rev. D **57**, 3351 (1998); J. Kijowski, Phys. Rev. A **59**, 897 (1999); A.D. Baute, R. Sala Mayato, J.P. Palao, J.G. Muga, and I.L. Egusquiza, *ibid.* **61**, 022118 (2000); K.-I. Aoki, A. Horikoshi, and E. Nakamura, *ibid.* **62**, 022101 (2000).

[4] J. Leon, J. Julve, P. Pitanga, and F.J. de Urries, Phys. Rev. A **61**, 062101 (2000); I.L. Egusquiza and J.G. Muga, *ibid.* **62**, 032103 (2000); J. Oppenheim, B. Reznik, and W. Unruh, J. Phys. A **35**, 7641 (2002); D. Alonso, R. Sala Mayato, and C.R. Leavens, Phys. Rev. A **66**, 042108 (2002); J.A. Damborenea, I.L. Egusquiza, G.C. Hegerfeldt, and J.G. Muga, *ibid.* **66**, 052104 (2002).

[5] N. Grot, C. Rovelli, and R.S. Tate, Phys. Rev. A **54**, 4676 (1996).

[6] V. Delgado and J.G. Muga, Phys. Rev. A **56**, 3425 (1997).

[7] V. Delgado, Phys. Rev. A **57**, 762 (1998); **59**, 1010 (1999).

[8] W.R. McKinnon and C.R. Leavens, Phys. Rev. A **51**, 2748 (1995); M. Daumer, in *Bohmian Mechanics and Quantum Theory: An Appraisal*, edited by J.T. Cushing, A. Fine, and S. Goldstein (Kluwer, Dordrecht, 1996), p. 87; C.R. Leavens, Phys. Rev. A **58**, 840 (1998); Y. Nogami, F.M. Toyama, and W. van Dijk, Phys. Lett. A **270**, 279 (2000); G. Gruebl and K. Rheinberger, J. Phys. A **35**, 2907 (2002).

[9] A. Challinor, A. Lasenby, S. Somaroo, C. Doran, and S. Gull, Phys. Lett. A **227**, 143 (1997).

[10] E. Deotto and G.C. Ghirardi, Found. Phys. **28**, 1 (1998).

[11] P. Holland, Found. Phys. **28**, 881 (1998).

[12] J. Finkelstein, Phys. Rev. A **59**, 3218 (1999).

[13] P. Holland, Phys. Rev. A **60**, 4326 (1999).

[14] C.R. Leavens, in *Time in Quantum Mechanics*, edited by J.G. Muga, R. Sala Mayato, and I.L. Egusquiza (Springer-Verlag, Berlin, 2002), p. 121.

[15] P. Holland, e-print quant-ph/0305175; see also, P. Holland and C. Philippidis, Phys. Rev. A **67**, 062105 (2003).

[16] N. Kemmer, Proc. R. Soc. London, Ser. A **173**, 91 (1939).

[17] W. Struyve and W. De Baere (private communication).