Optimal free will on one side in reproducing the singlet correlation

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Abstract
Bell's theorem teaches us that there are quantum correlations that cannot be simulated by just shared randomness (local hidden variable). There are some recent results which simulate the singlet correlation by using either 1 bit or a binary (no-signaling) correlation which violates Bell's inequality maximally. But there is one more possible way to simulate quantum correlation by relaxing the condition of independency of measurement on shared randomness. Recently, Hall showed that the statistics of a singlet state can be generated by sacrificing measurement independence where underlying distribution of hidden variables depends on measurement directions of both parties (Hall 2010 *Phys. Rev. Lett.* **105** 250404). He also proved that for any model of singlet correlation, 86% measurement independence is optimal. In this paper, we show that 59% measurement independence is optimal for simulating the singlet correlation when the underlying distribution of hidden variables depends only on the measurements of one party. We also show that a distribution corresponding to this optimal lack of free will already exists in the literature which first appeared in the context of detection efficiency loophole (Gisin and Gisin 1999 *Phys. Lett.* A 323–7).

1. Introduction

It has been found that certain quantum mechanical correlations violate some statistical inequalities which are based on assumptions of *realism*, *locality* and *measurement* *independence* (the experimenter's freedom to choose the measurement settings). These are commonly known as Bell-type inequalities [2] whose violation certifies non-locality present in such correlation. It is interesting to know which non-quantum resources are useful for simulating non-local correlation. The possible (non-quantum) resources to reproduce the quantum mechanical correlations are the following:

(i) classical communication aided by shared randomness,
(ii) some no-signaling non-local correlation (P-R box [3]) aided by shared randomness,
(iii) lack of free will on the part of parties performing local measurements [4, 5].

Toner–Bacon [6] have shown that the quantum correlation of a singlet state can be simulated by 1 bit communication (signaling correlation) where the measurement output is deterministic. Cerf *et al* [7] have also reproduced the correlation of a singlet state by using a binary input–output no-signaling correlation and in this case the measurement output is completely random. Recently, Hall [9] and Kar *et al* [8] independently conjectured a trade-off relation between the amount of signaling correlation (classical communication) and indeterminism for an output in simulating the singlet correlation. In all the above models, the assumption of measurement independence has been maintained. Later, in [4] Hall showed that the relaxation of measurement independence can be a useful resource for modeling the singlet state correlation. He provided a deterministic model of a singlet state in which only 14% measurement dependence is required.

In this paper, following the approach in [9], we address the question: how much measurement dependence is required to model the singlet state correlation and P-R correlation [3] if only one party gives up his/her experimental free will while the other party enjoys full experimental free will? Here, we have shown that 41% measurement dependence on one side is necessary for simulating the singlet correlation. Interestingly, there is a model which explicitly simulates the singlet correlation showing that 41% measurement dependence is also sufficient. By constructing some toy model, we also discuss the relation between the lack of free will and the amount of violation of Bell-CHSH inequality.

2. Lack of free will and modified Bell inequality

Consider a set of correlations $\{p(a, b|X, Y)\}$ of a binary input–output system, where X and a are respectively an input and an output on one side (Alice) and similarly Y and b for the other side (Bob). If there is an underlying variable λ in such a model which reproduces this correlation, then

$$p(a, b|X, Y) = \int \mathrm{d}\lambda \rho(\lambda|XY) p(a, b|X, Y, \lambda). \tag{1}$$

The measurement independence is a property that describes that the distribution of the underlying variable is independent of the measurement settings, i.e.

$$\rho(\lambda|X, Y) = \rho(\lambda|X', Y') \tag{2}$$

for any pair of joint settings (X, Y) and (X', Y'), which can be equivalently expressed as $\rho(\lambda|XY) = \rho(\lambda)$. The degree of measurement dependence is quantified by the variational distance between the distribution of the shared parameter for any pair of measurement settings [4, 10]:

$$M := \sup_{X,X',Y,Y'} \int \mathrm{d}\lambda \left|\rho(\lambda|XY) - \rho(\lambda|X'Y')\right|. \tag{3}$$

Clearly, the value of M can vary between 0 and 2.

The experimental free will, i.e. the fraction of measurement independence corresponding to a given model is defined as [4]

$$F := 1 - M/2. \tag{4}$$

For $M = 0$, $F = 1$ implies no lack of free will, i.e. it corresponds to the maximum possible degree of measurement independence (see equation (2)).

Now, let X, X' and Y, Y' denote possible measurement settings for Alice and Bob, respectively, and label each measurement outcome by 1 or -1. Let $\langle XY \rangle$ denote the average of the product of the measurement outcomes, for joint measurement settings X and Y. If for any correlation model in which the assumption of no communication, determinism and complete free will hold, then the correlation statistics satisfies the following so-called Bell-CHSH [11] inequality:

$$|\langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle - \langle X'Y'\rangle| \leqslant 2. \tag{5}$$

We now derive the following Bell-type inequality for any underlying deterministic no-signaling model in which measurement dependence for one side is invoked:

$$B = |\langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle - \langle X'Y'\rangle| \leqslant 2 + M, \tag{6}$$

where $0 \leqslant M \leqslant 2$.

Proof. Consider a deterministic hidden variable model in which the distribution of hidden variables depends on the choice of measurement settings. Then Bell-CHSH expression can be written as

$$\begin{aligned} B &= |\langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle - \langle X'Y'\rangle| \\ &= \left| \int [\rho(\lambda|XY)V(X|\lambda)V(Y|\lambda) + \rho(\lambda|XY')V(X|\lambda)V(Y'|\lambda) \right. \\ &\quad \left. + \rho(\lambda|X'Y)V(X'|\lambda)V(Y|\lambda) - \rho(\lambda|X'Y')V(X'|\lambda)V(Y'|\lambda]\mathrm{d}\lambda \right|. \end{aligned} \tag{7}$$

Since the model is deterministic, for any given λ, each observable takes a definite value, say $V(X|\lambda), V(X'|\lambda), V(Y|\lambda)$ and $V(Y'|\lambda)$ from the set $\{+1, -1\}$.

Now, for one-sided measurement-dependent models, without loss of generality, we assume that the distribution of λ depends only on Alice's measurements X and X'. Then, this condition can be expressed as

$$\begin{aligned} \rho(\lambda|XY) &= \rho(\lambda|XY') = \rho(\lambda|X) \\ \rho(\lambda|X'Y) &= \rho(\lambda|X'Y') = \rho(\lambda|X'). \end{aligned}$$

Under the above two conditions equation (7) now reduces to

$$\begin{aligned} B &= |\langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle - \langle X'Y'\rangle| \\ &= \left| \int [\rho(\lambda|X)V(X|\lambda)V(Y|\lambda) + \rho(\lambda|X)V(X|\lambda)V(Y'|\lambda) \right. \\ &\quad \left. + \rho(\lambda|X')V(X'|\lambda)V(Y|\lambda) - \rho(\lambda|X')V(X'|\lambda)V(Y'|\lambda]\, \mathrm{d}\lambda \right|. \end{aligned} \tag{8}$$

Next, since the absolute value of an integral is less than or equal to an integral of the absolute value of the integrand ($|\int f(x)\,\mathrm{d}x| \leqslant \int |f(x)|\,\mathrm{d}x$), from equation (8) we obtain

$$\begin{aligned} B &\leqslant \int |\rho(\lambda|X)V(X|\lambda)V(Y|\lambda) + \rho(\lambda|X)V(X|\lambda)V(Y'|\lambda) \\ &\quad + \rho(\lambda|X')V(X'|\lambda)V(Y|\lambda) - \rho(\lambda|X')V(X'|\lambda)V(Y'|\lambda)|\, \mathrm{d}\lambda \\ &= \int |V(Y|\lambda)\{\rho(\lambda|X)V(X|\lambda) + \rho(\lambda|X')V(X'|\lambda)\} \\ &\quad + V(Y'|\lambda)\{\rho(\lambda|X)V(X|\lambda) - \rho(\lambda|X')V(X'|\lambda)\}|\, \mathrm{d}\lambda. \end{aligned} \tag{9}$$

Table 1. Deterministic no-signaling model for correlations violating BI.

λ	$X(\lambda)$	$X'(\lambda)$	$Y(\lambda)$	$Y'(\lambda)$	$\rho(\lambda\|XY)$	$\rho(\lambda\|XY')$	$\rho(\lambda\|X'Y)$	$\rho(\lambda\|X'Y')$
λ_1	$-a$	$-a$	$-a$	a	0	0	p	p
λ_2	b	b	b	b	1	1	$1-p$	$1-p$

By applying the triangle inequality to the integrand, we obtain

$$\begin{aligned} B \leqslant & \int [|V(Y|\lambda)\{\rho(\lambda|X)V(X|\lambda) + \rho(\lambda|X')V(X'|\lambda)\}| \\ & + |V(Y'|\lambda)\{\rho(\lambda|X)V(X|\lambda) - \rho(\lambda|X')V(X'|\lambda)\}|]\, \mathrm{d}\lambda \\ = & \int \left[\left| V(X|\lambda)V(Y|\lambda) \left\{ \rho(\lambda|X) + \rho(\lambda|X') \frac{V(X'|\lambda)}{V(X|\lambda)} \right\} \right| \right. \\ & \left. + \left| V(X|\lambda)V(Y'|\lambda) \left\{ \rho(\lambda|X) - \rho(\lambda|X') \frac{V(X'|\lambda)}{V(X|\lambda)} \right\} \right| \right] \mathrm{d}\lambda. \end{aligned} \tag{10}$$

Since Alice's and Bob's observable attains a value ± 1,

$$B \leqslant \int [|\{\rho(\lambda|X) + k\rho(\lambda|X')\}| + |\{\rho(\lambda|X) - k\rho(\lambda|X')\}|]\, \mathrm{d}\lambda, \tag{11}$$

where $k = \frac{V(X'|\lambda)}{V(X|\lambda)}$, which for a given λ either takes a value $+1$ or -1. Then, the symmetry of the integrand on changing the value of k implies that

$$\begin{aligned} B &\leqslant \int |\rho(\lambda|X) + \rho(\lambda|X')|\, \mathrm{d}\lambda + \int |\rho(\lambda|X) - \rho(\lambda|X')|\, \mathrm{d}\lambda \\ &= 2 + \int |\rho(\lambda|X) - \rho(\lambda|X')|\, \mathrm{d}\lambda. \end{aligned} \tag{12}$$

By taking the supremum over Alice's measurements X and X', we obtain our desired result

$$B \leqslant 2 + \sup_{X,X'} \int |\rho(\lambda|X) - \rho(\lambda|X')|\, \mathrm{d}\lambda = 2 + M. \tag{13}$$

In a no-signaling deterministic model, the one-sided degree of measurement dependence that is required to reproduce the correlation corresponding to the value of the Bell expression $2\sqrt{2}$ is given by

$$M = 2(\sqrt{2} - 1).$$

The experimental free will for $B = 2\sqrt{2}$ is given by

$$F = 1 - M/2 = 2 - \sqrt{2} = 0.59. \tag{14}$$

For modeling P-R correlation by a no-signaling deterministic model, we need measurement dependence

$$\begin{aligned} M = &\ 2 \\ &\Rightarrow F = 0. \end{aligned}$$

So in this case there is no experimental free will for Bob.

Now, we provide a toy model with a tabular form which reproduces the above correlation and is also compatible with the above results. The basic assumptions of these models are no-signaling and determinism.

In table 1, we describe the singlet model which contains two underlying variables λ_1 and λ_2 with the outcome for the measurement setting X denoted by $X(\lambda_j)$. For this model, outcomes are specified by two numbers $a, b \in \{1, -1\}$. Here, the probability distribution is defined by a single parameter, $0 \leqslant p \leqslant 1$. From this table, $\langle XY\rangle = \langle X'Y\rangle = \langle XY'\rangle = 1$ and $\langle X'Y'\rangle = 1-2p$. Therefore, the value of the Bell expression is $B = \langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle - \langle X'Y'\rangle = 2+2p$; this violates Bell-CHSH by $2p$. The degree of measurement dependence is $M = 2p$ which clearly shows that $0 \leqslant M \leqslant 2$ and $0 \leqslant F \leqslant 1$. Therefore, for $B = 2 + M$, the amount of one-sided free will is $F = 1 - p$, in agreement with equation (14). So, here one can also see that the maximal Bell violation (PR correlation) can be simulated only by sacrificing the full free will of one party ($p = 1$). It is known that by sacrificing free will on both sides by 34% [4], the maximal Bell violation can be achieved. □

3. Lack of free will of one party and singlet simulation

For a binary input–output system, let X and Y denote Alice's and Bob's inputs. Alice and Bob share a random variable λ which is a real three-dimensional unit vector, distributed according to a distribution with probability density

$$\rho(\lambda|X, Y) = \rho(\lambda|X) = |X.\lambda/2\pi|, \tag{15}$$

i.e their underlying distribution depends on only one party's measurement direction. It has already been shown that the singlet correlation can be reproduced by sharing the above type of distribution [1]. For this distribution, the amount of measurement independence is given by

$$M := \sup_{X,X'} \int \mathrm{d}\lambda \, ||X.\lambda/2\pi| - |X'.\lambda/2\pi||. \tag{16}$$

The value of M can be computed, without loss of generality, if we choose a reference frame such as $X = (1, 0, 0)$, $X' = (\cos\beta, \sin\beta, 0)$ and $\lambda = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$,

$$M = \sup_{\beta} \frac{1}{4}\int_0^{2\pi} ||\cos\phi| - |\cos(\phi-\beta)|| \, \mathrm{d}\phi = 2\sqrt{2} - 2,$$

where the supremum value is attained for a particular setting $X' = (0, 1, 0)$.

So, the corresponding fraction of measurement independence is

$$F = 2 - \sqrt{2} = 0.59. \tag{17}$$

Note that there is another measure for the correlation between the hidden variable λ and measurement settings of Alice (X) and Bob (Y) provided in the model given by Barrett and Gisin [5]. In this model the degree of measurement dependence is quantified by mutual information $I(x, y : \lambda)$. For one-sided measurement dependence where the distribution of hidden variables is given by equation (15), it has been shown that $I(x, y : \lambda) = I(x : \lambda) \approx 0.28$ bits.

4. Conclusion

The main result of this paper is that one-sided measurement dependence can be used as a useful resource for simulating correlations that violate the Bell-CHSH inequality. Like the classical communication and no-signaling non-local resource, the lack of free will of one or both the parties can be used as a non-local resource. We find the optimal measurement independence for the first case in simulating singlet statistics and relate it to a distribution of hidden variables, which is a function of measurement direction of one party.

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References

[1] Gisin N and Gisin B 1999 *Phys. Lett.* A 323
[2] Bell J S 1964 *Physics* **1** 195
[3] Popescu S and Rohrlich D 1994 *Found. Phys.* **24** 379
[4] Hall M J W 2010 *Phys. Rev. Lett.* **105** 250404
[5] Barrett J and Gisin N 2011 *Phys. Rev. Lett.* **106** 100406
[6] Toner B F and Bacon D 2003 *Phys. Rev. Lett.* **91** 187904
[7] Cerf N J, Gisin N, Massar S and Popescu S 2005 *Phys. Rev. Lett.* **94** 220403
[8] Kar G *et al* 2011 *J. Phys. A: Math. Theor.* **44** 152002
[9] Hall M J W 2010 *Phys. Rev.* A **82** 062117
[10] Hall M J W 2011 *Phys Rev.* A **84** 022102
[11] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880