

Strong quantum violation of the gravitational weak equivalence principle by a non-Gaussian wave packet

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Abstract

The weak equivalence principle of gravity is examined at the quantum level in two ways. First, the position detection probabilities of particles described by a non-Gaussian wave packet projected upwards against gravity around the classical turning point and also around the point of initial projection are calculated. These probabilities exhibit mass dependence at both these points, thereby reflecting the quantum violation of the weak equivalence principle. Second, the mean arrival time of freely falling particles is calculated using the quantum probability current, which also turns out to be mass dependent. Such a mass dependence is shown to be enhanced by increasing the non-Gaussianity parameter of the wave packet, thus signifying a stronger violation of the weak equivalence principle through a greater departure from Gaussianity of the initial wave packet. The mass dependence of both the position detection probabilities and the mean arrival time vanishes in the limit of large mass. Thus, compatibility between the weak equivalence principle and quantum mechanics is recovered in the macroscopic limit of the latter. A selection of Bohm trajectories is exhibited to illustrate these features in the free fall case.

1. Introduction

In the famous gedanken experiment conceived by Galileo, the universality of the ratio between the gravitational and inertial masses has been studied with test bodies in free fall

from the leaning tower of Pisa [1]. Since then, several tests of weak equivalence principle have been performed with classical test bodies such as very sensitive pendula or torsion balances establishing extremely accurate conformation of the equality of gravitational and inertial masses. The most familiar tests of the weak equivalence principle are experiments of the Eötvös-type [2], which measure the gravitational acceleration of macroscopic objects. Thus, Einstein's weak equivalence principle is established in the Newtonian picture of the gravitational force which is exactly proportional to the inertial mass. The validity of the principle even for quantum mechanical particles is also proved using gravity-induced interference experiments [3, 4].

The traditional equivalence principle is fundamentally both classical and local, and it is interesting to enquire how it is to be understood in quantum mechanics. The principle of equivalence states that inertial mass is equal to gravitational mass: $m_i = m_g = m$. Another statement is that when all sufficiently small test bodies fall freely, they acquire an equal acceleration independent of their mass or constituent in a gravitational field. Quantum mechanically, it [5] can be said that 'The results of experiments in an external potential comprising just a sufficiently weak, homogeneous gravitational field, as determined by the wavefunction, are independent of the mass of the system'. The last statement is also known as the weak equivalence principle of quantum mechanics (WEQ).

The possibility of violation of weak equivalence principle in quantum mechanics is discussed in a number of papers, for instance, using neutrino mass oscillations in a gravitational potential [6]. The evidence of violation of WEQ can be shown both experimentally and theoretically. Experimental evidence exists in the interference phenomenon associated with the gravitational potential in neutron and atomic interferometry experiments [3, 4]. Theoretically, for a particle bound in an external gravitational potential, it is seen that the radii, frequencies and binding energy depend on the mass of the bound particle [7–9].

The free fall of test particles in a uniform gravitational field in the case of quantum states with and without a classical analogue was re-examined by Viola and Onofrio [10]. Another quantum mechanical approach of the violation of WEQ was given by Davies [11] for a quantum particle in a uniform gravitational field using a model quantum clock [12]. A quantum particle moving in a gravitational field can penetrate the classically forbidden region beyond the classical turning point of the gravitational potential and the tunnelling depth depends on the mass of the particle. So there is a small mass-dependent quantum delay in the return time representing a violation of weak equivalence principle. This violation is found within a distance of roughly one de Broglie wavelength from the turning point of the classical trajectory.

In the recent past, Ali *et al* [13] have shown an explicit mass dependence of the position probabilities for quantum particles projected upwards against gravity around both the classical turning point and the point of initial projection using Gaussian wave packet. They also have shown an explicit mass dependence of the mean arrival time at an arbitrary detector location for a Gaussian wave packet under free fall. Thus, both the position probabilities and the mean arrival time show the violation of WEQ for quantum particles represented by Gaussian wave packets. In this paper, using a class of non-Gaussian wave packets involving a tunable parameter, we will compute position detection probabilities around the classical turning point and the point of initial projection and also compute the mean arrival time for atomic and molecular mass particles represented by non-Gaussian wave packets to observe stronger violation of WEQ for quantum particles. To compute the mean arrival time, we consider the case when the particles are dropped from a height with zero initial velocity.

Violation of WEQ arises as a consequence of the spread of wave packets, the magnitude of which itself depends on the mass. In order to illustrate this effect, we present an analysis

in terms of Bohmian trajectories. In the Bohmian model [14], each individual particle is assumed to have a definite position, irrespective of any measurement. The pre-measured value of position is revealed when an actual measurement is done. Over an ensemble of particles having the same wavefunction ψ , these ontological positions are distributed according to the probability density $\rho = |\psi|^2$ where the wavefunction ψ evolves with time according to the Schrödinger equation and the equation of motion of any individual particle is determined by the guidance equation $v = J/\rho$, where v is the Bohmian velocity of the particle and J is the probability current density. Solving the guidance equation, one gets the trajectory of the particle. In this work, we describe the evolution of Bohmian trajectories in terms of the mass and the tunable parameter of the non-Gaussian wave packet.

2. Position detection probability

Let us consider an ensemble of quantum particles projected upwards against gravity with a given initial mean position and mean velocity. Our motivation is to look for greater violation of the equivalence principle compared to that achieved by a Gaussian wave packet. Hence, we formulate a one-dimensional non-Gaussian wavefunction which represents the initial state of each particle, given by

$$\psi(z, t = 0) = N \left[1 + \alpha \sin\left(\frac{\pi}{4\sigma_0} z\right) \right] e^{-\frac{z^2}{4\sigma_0^2} + ik_0 z}. \quad (1)$$

Here, α is the tuneable parameter varying from 0 to 1. The normalization factor N is given by

$$N = \left(\sqrt{2\pi}\sigma_0 \left[1 + \frac{\alpha^2}{2} (1 - e^{-\pi^2/8}) \right] \right)^{-1/2}. \quad (2)$$

The salient features corresponding to the above wave packet are its asymmetry, its infinite tail and its reducibility to a Gaussian wave packet upon a continuous decrement of the parameter α to zero. The property of an infinite tail of the above wavefunction is not associated with non-Gaussian forms that are generated by truncating the Gaussian distribution. The asymmetry of the wave packet (due to the sine function) entails a difference between its mean and peak, as will be further evident in our following analysis.

The initial group velocity is given by $u = \frac{\hbar k_0}{m}$, where m is the mass of the particle and k_0 is the wave number. For $\alpha = 0$, we get the Gaussian wavefunction. If ψ_1 and ψ_2 represent the initial wavefunctions for particles 1 and 2 in the Schrödinger picture, then the mean initial conditions are

$$\begin{aligned} \langle \hat{z} \rangle_{\psi_1} &= \langle \hat{z} \rangle_{\psi_2} = \langle \hat{z}(0) \rangle, \\ \frac{\langle \hat{p}_z \rangle_{\psi_1}}{m_1} &= \frac{\langle \hat{p}_z \rangle_{\psi_2}}{m_2} = u, \end{aligned} \quad (3)$$

where $\langle \hat{z} \rangle_{\psi}$ and $\langle \hat{p}_z \rangle_{\psi}$ represent the expectation values for position and momentum operators, respectively. Here, we have restricted ourselves to a one-dimensional representation along the vertical z direction for simplicity. With the propagator

$$G(z, t|y, 0) = \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{im}{2\hbar t}(z-y)^2 - \frac{img}{2\hbar}(z+y) - \frac{img^2 t^3}{24\hbar}} \quad (4)$$

of a particle in the linear gravitational potential $V = mgz$ and with the relation

$$\psi(z, t) = \int dy G(z, t|y, 0) \psi(y, 0), \quad (5)$$

one can get the Schrödinger time-evolved wavefunction $\psi(z, t)$ at any subsequent time t as

$$\psi(z, t) = N \sqrt{\frac{\sigma_0}{s_t}} e^{\frac{im}{2\hbar t} [z^2 - gt^2 z - \frac{g^2 t^4}{12}]} \left[e^{-\frac{im}{2\hbar t} \frac{\sigma_0}{s_t} [z - \frac{\hbar t}{m} k_0 + \frac{gt^2}{2}]^2} + \frac{\alpha}{2i} e^{-\frac{im}{2\hbar t} \frac{\sigma_0}{s_t} [z - \frac{\hbar t}{m} (k_0 + \beta) + \frac{gt^2}{2}]^2} - \frac{\alpha}{2i} e^{-\frac{im}{2\hbar t} \frac{\sigma_0}{s_t} [z - \frac{\hbar t}{m} (k_0 - \beta) + \frac{gt^2}{2}]^2} \right] \quad (6)$$

where $s_t = \sigma_0 \left(1 + i \frac{\hbar t}{2m\sigma_0^2}\right)$ and $\beta = \frac{\pi}{4\sigma_0}$. The probability density is given by

$$\rho(z, t) = |\psi(z, t)|^2 = N^2 \frac{\sigma_0}{\sigma_G} e^{E_1} \left[e^{E_2} + \alpha (e^{E_3} \sin A_1 + e^{E_4} \sin A_2) + \frac{\alpha^2}{4} (1 + e^{E_5} - 2e^{E_6} \cos A_3) \right], \quad (7)$$

where

$$\begin{aligned} E_1 &= -\frac{[\pi \hbar t + 4m\sigma_0 (z - ut + \frac{1}{2}gt^2)]^2}{32m^2\sigma_G^2\sigma_0^2}, \\ E_2 &= \frac{\pi \hbar t [\pi \hbar t + 8m\sigma_0 (z - ut + \frac{1}{2}gt^2)]}{32m^2\sigma_G^2\sigma_0^2}, \\ E_3 &= \frac{\pi \hbar t [\pi \hbar t + 24m\sigma_0 (z - ut + \frac{1}{2}gt^2)]}{64m^2\sigma_G^2\sigma_0^2}, \\ E_4 &= \frac{\pi \hbar t [\pi \hbar t + 8m\sigma_0 (z - ut + \frac{1}{2}gt^2)]}{64m^2\sigma_G^2\sigma_0^2}, \\ E_5 &= \frac{\pi \hbar t (z - ut + \frac{1}{2}gt^2)}{2m\sigma_G^2\sigma_0}, \\ E_6 &= \frac{E_5}{2}, \\ A_1 &= \frac{\pi [-\pi \hbar t + 8m\sigma_0 (z - ut + \frac{1}{2}gt^2)]}{32m\sigma_G^2}, \\ A_2 &= \frac{\pi [\pi \hbar t + 8m\sigma_0 (z - ut + \frac{1}{2}gt^2)]}{32m\sigma_G^2}, \\ A_3 &= \frac{\pi \sigma_0 (z - ut + \frac{1}{2}gt^2)}{2\sigma_G^2} \end{aligned}$$

and

$$\sigma_G = |s_t| = \sigma_0 \left(1 + \frac{\hbar^2 t^2}{4m^2\sigma_0^4}\right)^{\frac{1}{2}} \quad (8)$$

is the spreading of the Gaussian wave packet. The probability density is explicitly mass dependent. Here, the spreading of the wave packet is given by

$$\sigma_{NG} = \frac{\sqrt{\lambda^{(0)} + \lambda^{(2)}\alpha^2 + \lambda^{(4)}\alpha^4}}{4m\sigma_0 [2e^{\frac{\pi^2}{8}} + \alpha^2(e^{\frac{\pi^2}{8}} - 1)]} \quad (9)$$

where

$$\begin{aligned} \lambda^{(0)} &= 64 e^{\frac{\pi^2}{4}} m^2 \sigma_0^2 \sigma_G^2, \\ \lambda^{(2)} &= 8 e^{\frac{\pi^2}{8}} \pi^2 m^2 \sigma_0^4 (1 - 2e^{\frac{\pi^2}{16}}) + 64 e^{\frac{\pi^2}{8}} m^2 \sigma_0^2 \sigma_G^2 (-1 + e^{\frac{\pi^2}{8}}) + 2e^{\frac{\pi^2}{4}} \pi^2 \hbar^2 t^2, \\ \lambda^{(4)} &= (e^{\frac{\pi^2}{8}} - 1) [16m^2 \sigma_0^2 \sigma_G^2 (e^{\frac{\pi^2}{8}} - 1) + 4\pi^2 m^2 \sigma_0^4 + e^{\frac{\pi^2}{8}} \pi^2 \hbar^2 t^2], \end{aligned} \quad (10)$$

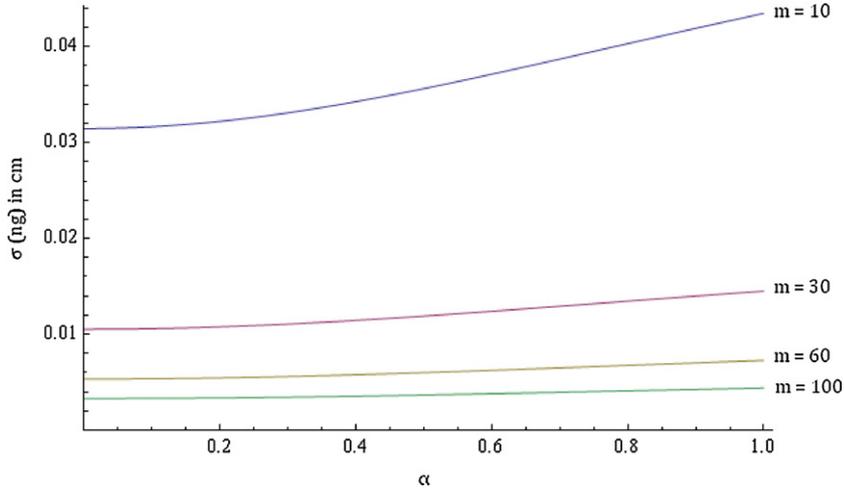


Figure 1. The variation of the width of the wave packet with α for different constant values of m (in amu). We take $\sigma_0 = 10^{-3}$ cm, $t = 1$ s and $u = 10^3$ cm s $^{-1}$.

which is also mass dependent.

Figure 1 gives the plot of σ_{NG} versus α . From this figure, we can see that for large masses, σ_{NG} becomes almost constant with α . For a particular value of α , σ_{NG} decreases with increasing value of mass m .

The expression for the expectation value of z is given by

$$\langle z \rangle = \frac{\pi \alpha \sigma_0}{2 + \alpha^2 (1 - e^{-\frac{\pi^2}{8}})} e^{-\frac{\pi^2}{32}} + ut - \frac{1}{2}gt^2. \quad (11)$$

Ehrenfest's theorem states that the expectation value of quantum mechanical operators obey classical laws of motion, the instances of which are given below:

$$\begin{aligned} \frac{d\langle z \rangle}{dt} &= \frac{\langle p \rangle}{m}, \\ \frac{d\langle p \rangle}{dt} &= \left\langle -\frac{\partial V}{\partial z} \right\rangle. \end{aligned} \quad (12)$$

It is possible to check that Ehrenfest's theorem is satisfied for our non-Gaussian wave packet. In the Gaussian limit (i.e. $\alpha \rightarrow 0$), $\langle z \rangle = z_{\text{peak}}$ and the dynamics of $\langle z \rangle$ is like a classical point particle, as expected.

In equation (11), the first term is α -dependent. There is no m in this expression. So $\langle z \rangle$ is α -dependent but mass independent. With the solutions obtained from equation $\frac{d|\psi|^2}{dz} = 0$, we can get the value of z_{peak} for which $|\psi|^2$ is maximum. But the equation cannot be solved analytically. So we have computed the z_{peak} numerically. z_{peak} depends both on α and mass. In table 1, it is shown numerically how z_{peak} and $\langle z \rangle$ vary with α for $0 \leq \alpha \leq 1$. In table 2, it is shown numerically how those two vary with mass. It is clear that $z_{\text{peak}} \neq \langle z \rangle$ and z_{peak} increases with α and mass m but $\langle z \rangle$ increases with α and remains constant for all masses.

Figure 2 shows the variation of z_{peak} and $\langle z \rangle$ with time simultaneously. The dotted curve shows the motion of z_{peak} whereas the continuous curve shows the motion of $\langle z \rangle$. As we take $\sigma_0 = 10^{-7}$ cm, the α -dependence of $\langle z \rangle$ and z_{peak} becomes vanishingly small. So this α -dependence cannot be shown graphically. As a result, the curves for z_{peak} and $\langle z \rangle$ coincide totally in this graph.

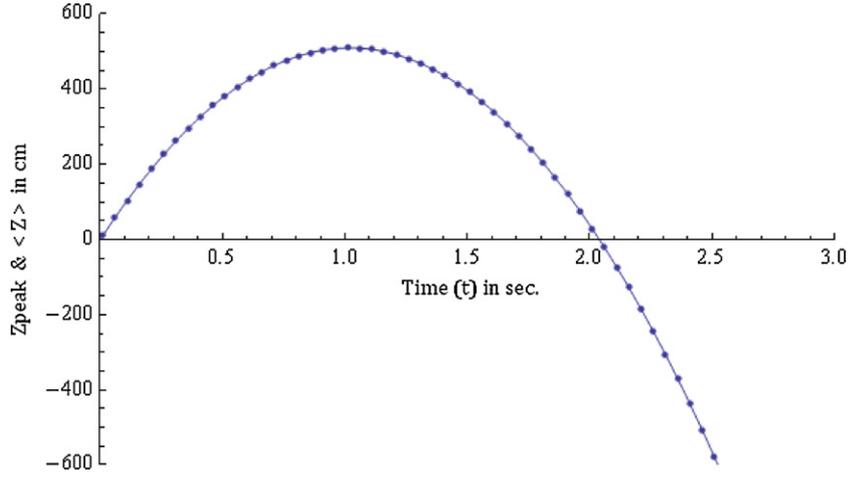


Figure 2. The variation of z_{peak} and $\langle z \rangle$ with time t . We take $\sigma_0 = 10^{-7}$ cm, $u = 10^3$ cm s $^{-1}$, $m = 50$ amu and $\alpha = 0.5$.

Table 1. The variation of $\langle z \rangle$ and z_{peak} in cm with α for mass $m = 10$ amu, $t = 2$ s and $\sigma_0 = 0.1$ cm.

α	z_{peak} (in cm)	$\langle z \rangle$ (in cm)
0	38.599 999 999 998 15	38.599 999 999 999 910
0.1	38.615 406 333 972 62	38.611 498 366 598 200
0.2	38.629 256 260 090 14	38.622 755 653 868 810
0.3	38.640 900 000 000 00	38.633 500 000 000 000
0.4	38.650 265 135 752 32	38.643 679 695 976 970
0.5	38.657 865 852 820 45	38.652 999 876 917 650
0.6	38.664 000 000 000 00	38.661 400 000 000 000
0.7	38.669 100 000 000 00	38.668 800 000 000 000
0.8	38.673 379 063 765 10	38.675 246 198 243 485
0.9	38.676 961 716 414 49	38.680 689 435 211 890
1.0	38.680 022 166 309 16	38.685 197 660 661 515

Table 2. The variation of $\langle z \rangle$ and z_{peak} in cm with mass m in amu for $\alpha = 0.5$, $t = 2$ s and $\sigma_0 = 0.1$ cm.

Mass (m) (in amu)	z_{peak} (in cm)	$\langle z \rangle$ (in cm)
30	38.657 865 898 827 980	38.652 999 876 917 65
60	38.657 865 903 136 155	38.652 999 876 917 65
90	38.657 865 903 934 166	38.652 999 876 917 65
120	38.657 865 904 213 070	38.652 999 876 917 65
150	38.657 865 904 342 295	38.652 999 876 917 65

Now, we want to observe the expression for $\langle z \rangle$ in the classical limit. If a full classical description is to emerge from quantum mechanics, we must be able to describe quantum systems in phase space. The Wigner distribution function [15, 16], which is one of the phase-space distributions, can provide a re-expression of quantum mechanics in terms of classical concepts.

The Wigner distribution function, which is calculated from the time-evolved wavefunction $\psi(z, t)$, is given by

$$D_w(z, p, t) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \psi^*(z+y, t) \psi(z-y, t) e^{\frac{2ipy}{\hbar}} dy \quad (13)$$

and the marginals of which yield the correct quantum probabilities for position and momentum separately.

By substituting the value of $\psi^*(z+y, t)$ and $\psi(z-y, t)$ using equation (5), we obtain the Wigner distribution function $D_w(z, p, t)$ for the non-Gaussian wave packet. It satisfies the classical Liouville's equation [17] given by

$$\frac{\partial D_w(z, p, t)}{\partial t} + \dot{z} \frac{\partial D_w(z, p, t)}{\partial z} + \dot{p} \frac{\partial D_w(z, p, t)}{\partial p} = 0. \quad (14)$$

Here, $\dot{z} = p/m$ and $\dot{p} = -mg$ are obtained from Hamilton's equation of motion given by

$$\begin{aligned} \dot{z} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial z}, \end{aligned} \quad (15)$$

where $H = p^2/2m + mgz$ is the Hamiltonian of the system.

The corresponding position distribution function is given by

$$\rho_C(z, t) = \int_{-\infty}^{\infty} D_w(z, p, t) dp. \quad (16)$$

So the expression for the average value of z using the Wigner distribution function is given by

$$\begin{aligned} \langle z \rangle &= \frac{\int_{-\infty}^{\infty} z \rho_C(z, t) dz}{\int_{-\infty}^{\infty} \rho_C(z, t) dz} \\ &= \frac{\pi\alpha\sigma_0}{2 + \alpha^2(1 - e^{-\frac{\pi^2}{8}})} e^{-\frac{\pi^2}{32}} + ut - \frac{1}{2}gt^2 \equiv z_0 + ut - \frac{1}{2}gt^2 \end{aligned} \quad (17)$$

which is exactly equal to the quantum mechanical expectation value of z .

The particles are projected upwards against gravity. Classically, the reference point $z = 0$ reaches the maximum height $z = z_{\max} = ut_1 - \frac{1}{2}gt_1^2$ at the time $t = t_1 = u/g$. So the probability of finding the particles $P_1(m)$ within a very narrow detector region $(-\epsilon to + \epsilon)$ around z_{\max} is given by

$$P_1(m) = \int_{z_{\max}-\epsilon}^{z_{\max}+\epsilon} |\psi(z, t_1)|^2 dz. \quad (18)$$

At a later time $t = t_2 = 2u/g$, the classical peak of the wave packet returns to its initial projection point at $z = 0$. Similarly, the probability of finding the particles $P_2(m)$ within a very narrow detector region $(-\epsilon to + \epsilon)$ around $z = 0$ is given by

$$P_2(m) = \int_{-\epsilon}^{+\epsilon} |\psi(z, t_2)|^2 dz. \quad (19)$$

From figures 3 and 4, it is clear that both the probabilities are mass dependent for an initial non-Gaussian position distribution for smaller masses. This effect of mass dependence of the probabilities occur essentially as the spreading of the wave packet under linear gravitational potential depends on the mass of the particles. These probabilities become saturated in the limit of large mass. As α increases, both the probabilities decrease for a particular value of m . The departure from the result for the Gaussian ($\alpha = 0$) case increases with the value of the non-Gaussian parameter, signifying a stronger violation of the weak equivalence principle.

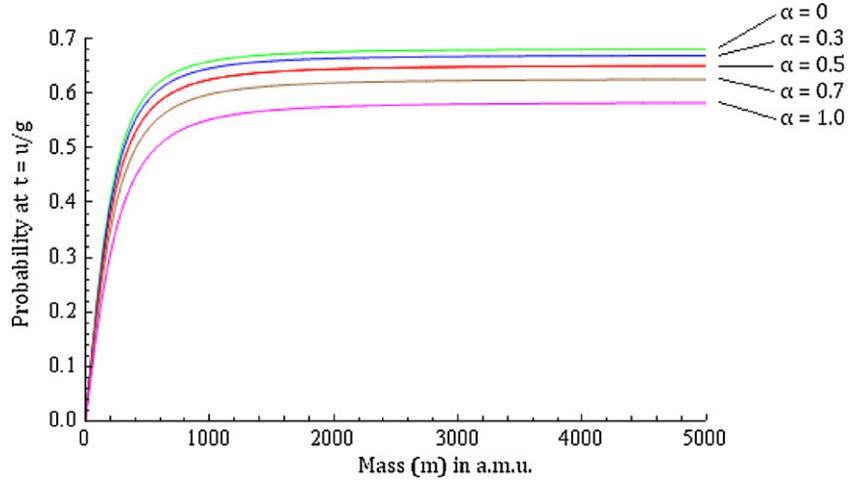


Figure 3. The variation of probability $P_1(m)$ with mass (in amu) for different constant values of α . We take $u = 10^3 \text{ cm s}^{-1}$, $\sigma_0 = 10^{-3} \text{ cm}$ and $\epsilon = \sigma_0$.

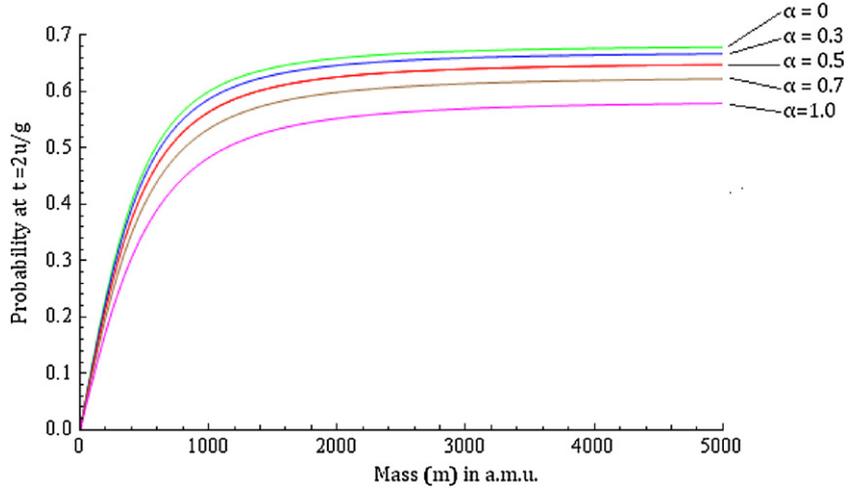


Figure 4. The variation of probability $P_2(m)$ with mass (in amu) for different constant values of α . We take $u = 10^3 \text{ cm s}^{-1}$, $\sigma_0 = 10^{-3} \text{ cm}$ and $\epsilon = \sigma_0$.

3. Mean arrival time

Now, let us consider the quantum particles subjected to the free fall from the initial position at $z = 0$ under gravity with the initial state given by equation (1) and with $u = 0$. Using the probability current approach [18], the mean arrival time of the quantum particles to reach a detector location at $z = Z$ is given by

$$\bar{\tau} = \frac{\int_0^\infty |J(z = Z, t)| t dt}{\int_0^\infty |J(z = Z, t)| dt} \quad , \quad (20)$$

where $J(z, t)$ is the quantum probability current density. However, we emphasize that the definition of the mean arrival time used in the above equation is not a uniquely derivable result within standard quantum mechanics. It should also be noted that $J(z, t)$ can be negative, so we take the modulus sign in order to use the above definition. From equation (20), it can be seen that the integral of the denominator converges whereas the integral of the numerator diverges formally. To avoid this problem, several techniques have been employed [19]. In this paper, to obtain convergent results of the numerator, we use the simple technique of getting a cut-off at $t = T$ in the upper limit of the time integral with $T = \sqrt{2(|Z| + 3\sigma_T)}/g$. σ_T is the width of the wave packet at time T . Thus, our calculations of the arrival time are valid up to the $3\sigma_T$ level of spread in the wavefunction.

Here, the expression for the Schrödinger probability current density $J(z, t)$ with the initial non-Gaussian position distribution is given by

$$J(z, t) = \frac{\hbar}{m} \Im(\psi^* \nabla \psi) = N^2 \frac{e^{E_1}}{32m^2\sigma_0\sigma_G^3} (\eta^{(0)} + \alpha\eta^{(1)} + \alpha^2\eta^{(2)}), \quad (21)$$

where

$$\begin{aligned} \eta^{(0)} &= 8 e^{E_2} \left[\hbar^2 t \left(z - \frac{1}{2} g t^2 \right) + 4m^2 \sigma_0^4 (u - g t) \right], \\ \eta^{(1)} &= \eta^{(0)} e^{-E_2} (e^{E_3} \sin A_1 + e^{E_4} \sin A_2) - 2\pi \hbar^2 t \sigma_0 (e^{E_3} \cos A_1 + e^{E_4} \cos A_2) \\ &\quad + 4\pi m \hbar \sigma_0^3 (e^{E_3} \sin A_1 - e^{E_4} \sin A_2), \\ \eta^{(2)} &= \left[2\hbar^2 t \left(z - \frac{1}{2} g t^2 \right) + 8m^2 \sigma_0^4 (u - g t) \right] (1 + e^{E_5} - 2 e^{E_6} \cos A_3) \\ &\quad - 2\pi \hbar^2 t \sigma_0 e^{E_6} \sin A_3 + 2\pi m \hbar \sigma_0^3 (-1 + e^{E_5}), \end{aligned}$$

and where E_i 's are defined below equation (7).

It is clear that $J(z, t)$ is explicitly mass dependent. So $\bar{\tau}$ is also mass dependent. From figure 5, we can observe that for smaller masses $\bar{\tau}$ is mass dependent and for larger masses it becomes mass independent. For a particular value of mass, $\bar{\tau}$ increases with α . From figure 6, we can see that for large masses, $\bar{\tau}$ becomes almost constant with α . For a particular value of α , $\bar{\tau}$ decreases with increasing value of mass m . It is clear that the mass dependence of the mean arrival time gradually vanishes as the mass is increased. Thus, compatibility with the equivalence principle emerges in the limit of large mass or classical limit. Note that with the increase of the parameter α , one needs to go to higher values of mass in order for the mass dependence to become insignificant. Thus, a non-Gaussian wave packet can exhibit non-classical features in a mass range where its Gaussian counterpart behaves classically.

4. Bohm trajectory approach

Within the Bohmian interpretation of quantum mechanics [14], each individual particle is assumed to have a definite position, irrespective of any measurement. The equation of motion of the particle is

$$m\ddot{\vec{x}} = -\nabla(V + Q)|_{\vec{x}=\vec{x}(t)}, \quad (22)$$

where $Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}$ is called 'quantum potential energy' and it is explicitly mass-dependent and $\vec{x}(t)$ determines the path of the particle. In the gravitational potential $V = mgz$, the

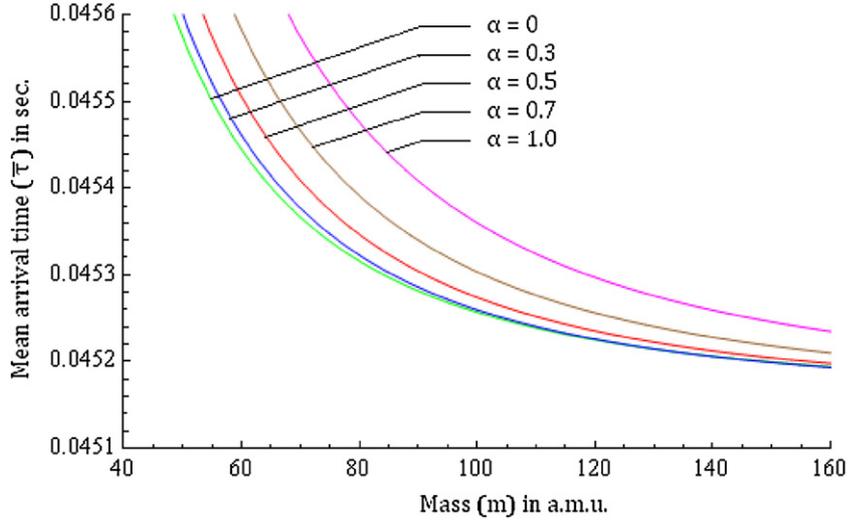


Figure 5. The variation of the mean arrival time with mass (in amu) for different constant values of α . We take $\sigma_0 = 10^{-6}$ cm, $Z = -1$ cm and $u = 0$.

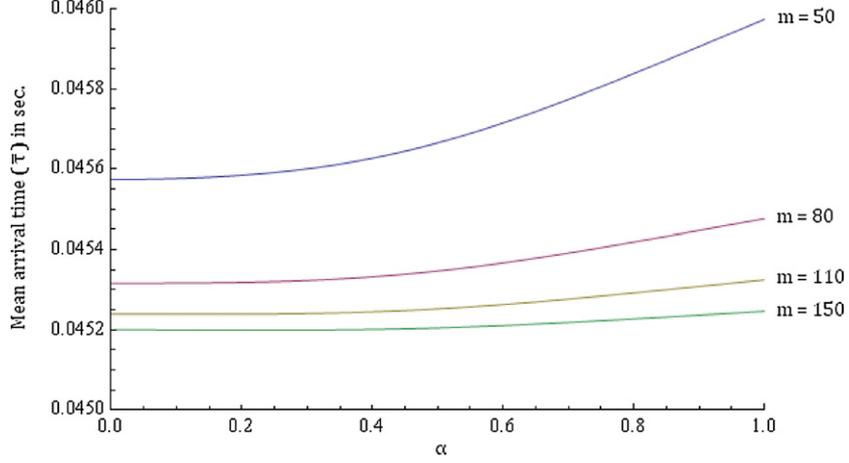


Figure 6. The variation of mean arrival time with α for different constant values of m (in amu). We take $\sigma_0 = 10^{-6}$ cm, $Z = -1$ cm and $u = 0$.

quantum version of Newton's second law is given by

$$\ddot{z} = g - \frac{1}{m} \frac{\partial Q}{\partial z} \Big|_{z=z(t)}. \quad (23)$$

To quote from Holland [5], 'It is immediately clear that the assumption of the equality of the inertial and passive gravitational masses does not imply that all bodies fall with an equal acceleration in a given gravitational field, due to the intervention of the mass-dependent quantum force term'. Thus, WEQ is violated. The arrival-time problem is unambiguously solved in the Bohmian mechanics, where for an arbitrary scattering potential $V(\vec{x})$, one finds [20] that for those particles that actually reach $\vec{x} = \vec{X}$, the arrival-time distribution is given by the modulus of the probability current density, i.e. $|J(\vec{X}, T)|$.

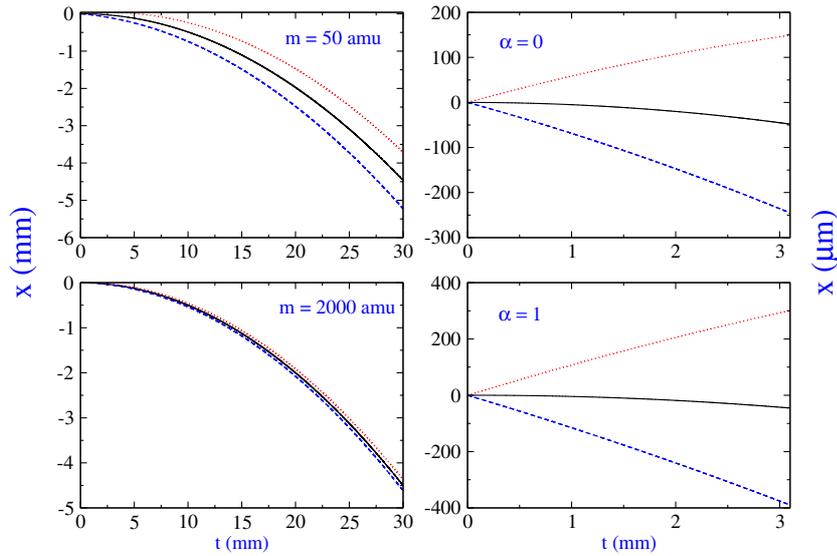


Figure 7. A selection of Bohmian trajectories for $u = 0$ (free fall) and $\sigma_0 = 10^{-6}$ cm. In the left column, the tunable parameter α has the fixed value $\alpha = 0.5$ and in the right one mass is constant and equal to $m = 100$ amu. The solid black trajectory starts at $\langle z \rangle(0)$, the dashed blue one at $\langle z \rangle(0) - 2\sigma_0$ and the dotted red one at $\langle z \rangle(0) + 2\sigma_0$.

The quantum potential Q is nonlocal, and thus it is expected that its effect on the violation of WEQ is manifested in a way similar to the violation observed as a consequence of the spread of the wave packet. We compute a set of Bohmian trajectories corresponding to the free fall of our non-Gaussian wave packet with the tunable parameter α . Figure 7 shows a selection of Bohmian trajectories exhibiting their spread depending upon the values of m and α . Note that for small mass, the trajectories with initial positions on the left and right of the centre of the wave packet (mean $\langle z \rangle$) spread out with evolution, indicating violation of WEQ. The magnitude of spread increases with α , signifying stronger violation of WEQ with increased departure from Gaussianity. The spread of the trajectories decreases as mass is increased, leading to the emergence of WEQ in the classical limit.

5. Conclusions

To summarize, in this work we have studied the violation of the gravitational weak equivalence principle in quantum mechanics using a non-Gaussian wave packet. The wave packet is constructed in such a way that its departure from Gaussianity is represented by a continuous and tunable parameter α . The asymmetry of the wave packet entails a difference between its mean and peak, and causes the peak to evolve differently from a classical point particle, whereas the mean evolves like a classical point particle modulo a constant depending upon the parameter α . Such a result is consistent with the Ehrenfest theorem, as we have shown, and re-confirmed using the Wigner distribution. The correspondence with the results following from a Gaussian wave packet is achieved in the limit of $\alpha \rightarrow 0$.

The weak equivalence principle is shown to be violated first through the dependence on mass of the position detection probabilities of freely falling particles. The magnitude of violation increases with the increase of the non-Gaussian parameter α , signifying stronger

violation of WEQ with larger deviation from Gaussianity. We next compute the arrival time of freely falling wave packets through the probability current distribution corresponding to the non-Gaussian wave packet. The violation of WEQ is again exhibited with the dependence on mass of the mean arrival time. It is observed that in the limit of large mass the classical value for the mean arrival time is approached, thereby indicating the emergence of the WEQ in the classical limit. In this context, it is worthwhile mentioning that though our work follows as a natural consequence of quantum mechanics or quantum theory in the non-relativistic limit, it does not imply a violation of general covariance in the relativistic domain. An interesting connection between the non-relativistic limit of quantum theory and the principle of equivalence has recently been discussed [21]. Finally, a computation of Bohmian trajectories further establishes the stronger violation of WEQ by a non-Gaussian wave packet. We conclude by noting that the realization of exactly Gaussian wave packets is rather difficult in real experiments. One of the consequences of our results is to enable relaxation of the wave packet to be Gaussian, and therefore it can facilitate the experimental observation of the violation of WEQ, as well as enable a quantitative verification of the way the violation of WEQ depends on the departure from the Gaussian nature of the wave packet. Other ramifications of this result call for further study.

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