Quantum teleportation using non-orthogonal entangled channels

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Abstract

We study quantum teleportation with the resource of non-orthogonal qubit states. We first extend the standard teleportation protocol to the case of such states. We investigate how the loss of teleportation fidelity resulting from the use of non-orthogonal states compares to a similar loss of fidelity when noisy or non-maximally entangled states are used as the teleportation resource. Our analysis leads to some interesting results on the teleportation efficiency of both pure and mixed non-orthogonal states compared to that of non-maximally entangled and mixed states.

1. Introduction

Quantum teleportation is an important and vital quantum information processing task where an arbitrary unknown quantum state can be replicated at a distant location using previously shared entanglement and classical communication between the sender and the receiver. A remarkable application of entangled states having many ramifications in information technology, quantum teleportation can also be combined with other operations in order to construct advanced quantum circuits useful for information processing [1]. The original idea of teleportation introduced by Bennett *et al* [2] is implemented through a channel involving a pair of particles in a Bell state shared by the sender and the receiver and at the end of the protocol an unknown input state is reconstructed with perfect fidelity at another location while destroying the original copy.

To implement teleportation, one ideally needs maximally entangled two-qubit states, i.e. singlet states. But in a typical experiment it is very difficult to prepare singlet states because the preparation is never perfect. As a result, one may generally have to deal with non-maximally entangled states or mixed states. So it becomes necessary to generalize the idea of using the maximally entangled states as quantum channels to the case of non-maximally entangled or noisy channels between two distant partners [3]. To switch over from a maximally entangled state to a non-maximally entangled state or a mixed state, one has to pay a price in terms of the loss of teleportation fidelity. In the case of mixed states, there are several works in the literature on schemes of teleportation using different categories of mixed states (see, e.g., the works [4, 5] for Werner states [6], the work [7] for a different class of mixed states [8] and the work [9] for teleportation using lossy channels). For non-maximally entangled channels the loss of fidelity of teleportation could be compensated for by schemes for probabilistic teleportation [12, 13]. Probabilistic teleportation schemes have been developed to teleport N qubits using directly N non-maximally entangled channels [14].

In quantum information theory, the entanglement of orthogonal states has received much attention. However, non-orthogonal states could also act as potentially useful resources for information processing. Examples of entangled non-orthogonal states are readily available; entangled

coherent states fall under this category. A Schrödinger cat state which has been proposed using $SU(2)$ coherent states forms a particular realization of the entanglement of non-orthogonal $SU(2)$ coherent states [15]. The theory of non-orthogonal quantum states has developed with the study of the Schmidt decomposition for a two-particle system involving non-orthogonal states [16]. The use of non-orthogonal quantum states for cryptographic purposes [17] is appreciated because non-orthogonal quantum states cannot be distinguished with perfect reliability [18]. Any attempt to do so (even imperfectly) imparts a disturbance to them [19]. Studies on non-orthogonal quantum states which form an overcomplete basis in the context of teleportation have been undertaken and a realistic protocol for the continuous variable teleportation of a coherent state has been conducted [20]. The continuous-variable teleportation protocol was implemented in an experiment that teleported a coherent state of an optical-frequency electromagnetic mode with fidelity 0.58 ± 0.02 [21]. Two other experiments have improved the experimental fidelity of the teleported coherent state to values of 0.64 ± 0.02 [22] and 0.61 ± 0.02 [23]. This protocol was developed further recently [24].

Given that teleportation protocols have been proposed with non-maximally entangled channels, mixed channels and non-orthogonal entangled channels, a systematic study is in order for evaluating their comparative performance in terms of their respective teleportation fidelities. With such motivation, in this work we first reformulate the original quantum teleportation protocol [2] in terms of non-orthogonal quantum states in a two-dimensional (2D) Hilbert space. Since the introduction of an infinitesimal amount of non-orthogonality decreases the amount of entanglement in a bipartite system, it would be interesting to see how the introduction of non-orthogonality in the system affects the fidelity of teleportation. The issue as to whether the average teleportation fidelity can be made to increase when a non-orthogonal two-qubit entangled system is used compared to the case when either a non-maximally orthogonal entangled state or a mixed state is used as a teleportation channel could be important for practical purposes. Further, since even among mixed states there exist two distinct categories, namely maximally entangled [6, 8] or non-maximally entangled [7, 25] states, a similar comparison could be made of a mixed entangled non-orthogonal channel with respect to a non-maximally entangled mixed state that has been found to yield a better teleportation fidelity than some other mixed states [25].

The plan of this paper is as follows. In the next section, we present our protocol for teleportation using a non-orthogonal qubit state as the channel. In section 3, we compare the performance (fidelity of teleportation) of this non-orthogonal channel with that of a non-maximally entangled channel, and also the Werner state used as a teleportation channel. We next conduct a similar study for the case of mixed states in section 4. Here we compute the fidelity of teleportation for a non-orthogonal mixed state and compare it with that of a non-maximally entangled mixed state. A summary of our results is presented in section 5.

2. Non-orthogonal entangled state and teleportation

A bipartite entangled state can, in general, be written as

$$|\alpha\rangle^{AB} = \mu|\alpha\rangle^A|\beta\rangle^B + \nu|\gamma\rangle^A|\delta\rangle^B, \qquad (1)$$

where the state vectors $|\alpha\rangle^A$ and $|\gamma\rangle^A$ for system A and $|\beta\rangle^B$ and $|\delta\rangle^B$ for system B represent the linearly independent non-orthogonal states that span a 2D subspace of each Hilbert space. The parameters μ and ν are complex coefficients. A bipartite entangled state involving non-orthogonal states would have the property that the overlaps $\langle\alpha|\gamma\rangle^A$ and $\langle\delta|\beta\rangle^B$ are non-zero.

Let us consider the 2D subspace of the Hilbert spaces H_A and H_B spanned by the linearly independent non-orthogonal quantum states $|\alpha\rangle$ and $|\beta\rangle$. The quantum states $|\alpha\rangle$ and $|\beta\rangle$ are non-orthogonal in the sense that their inner product is non-vanishing, i.e. $\langle\alpha|\beta\rangle \neq 0$. Since, in general, $\langle\alpha|\beta\rangle$ is complex, we can assume that $\langle\alpha|\beta\rangle = re^{i\theta}$, where the real parameters r and θ, respectively, denote the modulus and argument of the complex number. Let us choose the normalized non-orthogonal basis vectors $|\alpha\rangle$ and $|\beta\rangle$ as

$$|\alpha\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\beta\rangle = \begin{pmatrix} re^{i\theta} \\ N_\beta \end{pmatrix}, \qquad (2)$$

where $N_\beta = \sqrt{1-r^2}$. Using the Gram–Schmidt ortho-normalization procedure, we transform non-orthogonal basis vectors $|\alpha\rangle$ and $|\beta\rangle$ into normalized orthogonal basis vectors $|0\rangle$ and $|1\rangle$ as

$$|0\rangle = |\alpha\rangle, \quad |1\rangle = N_G[|\beta\rangle - \langle\alpha|\beta\rangle|\alpha\rangle], \qquad (3)$$

where $N_G = \frac{1}{N_\beta} = \frac{1}{\sqrt{1-r^2}}$. Here we note that the state $|1\rangle$ contains implicitly information on non-orthogonality of the original non-orthogonal system. So we may now proceed with usual orthogonal basis vectors $|0\rangle$ and $|1\rangle$.

A bipartite entangled state in the non-orthogonal basis can be written as

$$|\Psi\rangle_{ab} = N_1[|\alpha\rangle|\beta\rangle + |\beta\rangle|\alpha\rangle] \qquad (4)$$

where $N_1 = \frac{1}{\sqrt{2(1+r^2)}}$ is the normalization constant. In terms of the orthogonal basis vectors $|0\rangle$ and $|1\rangle$, equation (4) can be re-expressed as

$$|\Psi\rangle_{ab} = \frac{N_1}{N_G}(|0\rangle|1\rangle + |1\rangle|0\rangle) + 2N_1\langle\alpha|\beta\rangle|0\rangle|0\rangle. \qquad (5)$$

In general, the concurrence for an arbitrary two-qubit pure state $a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is given by $2|ad - bc|$ [10].

Therefore, the amount of entanglement contained in the bipartite entangled state $|\Psi\rangle_{ab}$ can be quantized by the concurrence (C) given by

$$C(|\Psi\rangle_{ab}) = \frac{1-r^2}{1+r^2}, \quad 0 \leqslant r \leqslant 1. \qquad (6)$$

For the pure state, the concurrence and negativity are equal [11], so

$$C(|\Psi\rangle_{ab}) = \mathcal{N}(|\Psi\rangle_{ab}) = \frac{1-r^2}{1+r^2}, \quad 0 \leqslant r \leqslant 1. \qquad (7)$$

The parameter r is a measure of the non-orthogonality. From expression (7), we find that C is a decreasing function of r and hence as the amount of non-orthogonality increases, the amount of entanglement in a bipartite system decreases and it goes to zero when the non-orthogonal parameter r tends to 1. The maximum amount of entanglement is achieved when $r = 0$, i.e. when the basis state vectors are orthogonal to each other.

Let us now formulate the teleportation protocol between two particles Alice and Bob, with the input state prepared by a third party Charlie. He sends the prepared state to Alice. In this transmission we assume that there is no distortion of the input state. Since the input state is given by the third party, Alice has no knowledge of the received state that she wants to teleport, and this arbitrary state is given by

$$|\phi\rangle_1 = x|0\rangle + y|1\rangle, \qquad (8)$$

where $x^2 + y^2 = 1$. Let us assume that two distant partners Alice and Bob share a two-qubit entangled state $|\Psi\rangle_{ab}$ given by equation (5). The particles a and b are with Alice and Bob, respectively. The state $|\Psi\rangle_{ab}$ acts as a quantum teleportation channel whose entanglement depends on how much non-orthogonality one introduces in the system.

Next, we combine the single qubit 1 and the two qubits a and b using the tensor product and then express the resulting three-qubit system as a tensor product of a single qubit b and the Bell basis involving the two qubits 1 and a, as

$$
\begin{aligned}
|\chi\rangle_{1ab} &= |\phi\rangle_1 \otimes |\Psi\rangle_{ab} \\
&= \frac{1}{\sqrt{2}}[|\Phi^+\rangle_{1a}(P_+|0\rangle_b + A|1\rangle_b) + |\Phi^-\rangle_{1a}(P_-|0\rangle_b + A|1\rangle_b) \\
&\quad + |\Psi^+\rangle_{1a}(Q_+|0\rangle_b + B|1\rangle_b) + |\Psi^-\rangle_{1a}(Q_-|0\rangle_b - B|1\rangle_b)],
\end{aligned}
\qquad (9)
$$

where

$$
A = \frac{xN_1}{N_G}, \quad B = \frac{yN_1}{N_G}, \quad P_\pm = N_1\left(2xre^{i\theta} \pm \frac{y}{N_G}\right)
$$

$$
Q_\pm = N_1\left(\frac{x}{N_G} \pm 2yre^{i\theta}\right) \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)
$$

$$
|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).
$$

Since the qubits 1 and a are with Alice, she makes a Bell state measurement on her qubits and then sends the measurement result to Bob expending two classical bits. According to the received measurement result $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$ or $|\Psi^-\rangle$, Bob performs a suitable unitary operation on his qubit b as follows:

1. If the measurement result is $|\Phi^+\rangle$, Bob operates σ_x on his qubit.
2. If the measurement result is $|\Phi^-\rangle$, Bob operates σ_y on his qubit.
3. If the measurement result is $|\Psi^+\rangle$, Bob operates the Identity operator I (does nothing) on his qubit.
4. If the measurement result is $|\Psi^-\rangle$, Bob operates σ_z on his qubit.

Thereafter Bob transmits this state to the third party Charlie whose task is to measure the efficiency of the teleportation protocol.

The teleportation fidelity is defined as [11]

$$F^{\text{tel}} = \sum_{j=1}^{4} P_j |\langle\phi|\xi_j\rangle|^2, \qquad (10)$$

where $|\phi\rangle$ is the input state and $|\xi_j\rangle$ $(j = 1, 2, 3, 4)$ are the normalized single-qubit output states after the unitary transformations and $P_j = \text{Tr}(M_j|\xi_j\rangle\langle\xi_j|)$ denotes the corresponding probability of obtaining the normalized output state $|\xi_j\rangle$, where $M_1 = |\Phi^+\rangle\langle\Phi^+|$, $M_2 = |\Phi^-\rangle\langle\Phi^-|$, $M_3 = |\Psi^+\rangle\langle\Psi^+|$, $M_4 = |\Psi^-\rangle\langle\Psi^-|$. In this case, the teleportation fidelity is found to be

$$F^{\text{tel}} = \frac{1 - r^2(1 - 2y^2)^2}{1 + r^2}. \qquad (11)$$

From equation (11), one can observe the following points:

1. If $y \to 0$ or $y \to 1$, the fidelity of teleportation of a qubit (lying in the neighborhood of classical bit) via the non-orthogonal entangled state $|\Psi\rangle_{ab}$ as a teleportation channel is given by

$$F^{\text{tel}} \to \frac{1 - r^2}{1 + r^2} = C(|\Psi\rangle_{ab}). \qquad (12)$$

The fidelity of teleportation (12) exceeds the classical fidelity $\frac{2}{3}$ when the parameter r satisfies the inequality $0 \leqslant r < \frac{1}{\sqrt{5}}$.

2. If $y = \frac{1}{\sqrt{2}}$, i.e. if the qubit is in an equal superposition of two classical bits, the fidelity of teleportation is given by

$$F_1^{\text{tel}} = \frac{1}{1 + r^2}. \qquad (13)$$

In this case, the fidelity (13) overtakes classical fidelity $\frac{2}{3}$ when $0 \leqslant r < \frac{1}{\sqrt{2}}$.

Since the teleportation fidelity (11) is input state dependent, it would be better to calculate the average fidelity over all input states. The average teleportation fidelity over all input states is given by

$$F_{\text{av}}^{\text{tel}} = \frac{3 - r^2}{3(1 + r^2)}. \qquad (14)$$

Here we have obtained the effect of non-orthogonality on the average teleportation fidelity when a non-orthogonal entangled quantum channel is used in the teleportation protocol. Let us now compare the results of this section with the case when a non-maximally entangled state is used as a teleportation channel.

3. Comparison of teleportation fidelities of a non-orthogonal channel with mixed and non-maximally entangled channels

One of the best known and perhaps also the simplest example of a mixed qubit state is the Werner state [6], which is a convex combination of a pure maximally entangled state and a
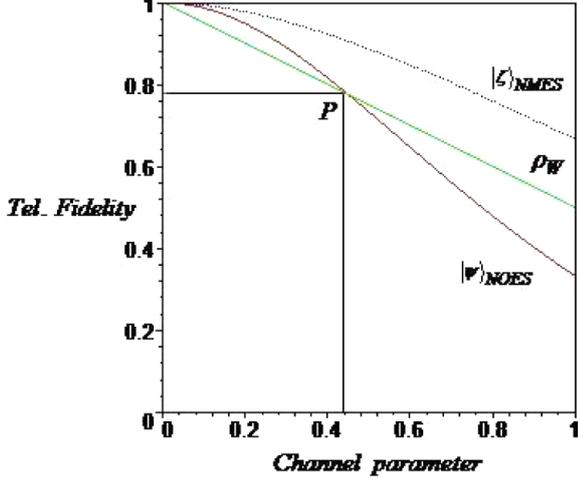
Figure 1. The teleportation fidelities of the three channels, namely the non-orthogonal state $|\psi\rangle_{\text{NOES}}$, the (mixed) Werner state ρ^W and the non-maximally entangled pure state $|\xi\rangle_{\text{NMES}}$, are plotted against their respective channel parameters r, p and s for comparison. The teleportation efficiency of $|\psi\rangle_{\text{NOES}}$ is better than that of to ρ^W in the parameter range where the classical limit of $2/3$ is exceeded. However, $|\xi\rangle_{\text{NMES}}$ outperforms the other two throughout.

maximally mixed state. Since the entanglement of the Werner state cannot be increased by any unitary transformation, it can be regarded as a maximally entangled mixed state [7]. The efficiency of the Werner state as a teleportation channel has been studied in detail in [5]. Let us briefly recapitulate here some of the essential features relevant to the present study. The Werner state can be expressed as

$$\rho^W = (1 - p)|\psi^-\rangle\langle\psi^-| + \frac{p}{4}I, \qquad (15)$$

where $|\psi^-\rangle$ is the singlet state and the parameter p lies between 0 and 1, with $p = 0$ denoting a maximally entangled pure state, while $p = 1$ denotes the maximally mixed state. When the Werner state is used as a teleportation channel, the average teleportation fidelity is given in terms of the parameter p as

$$W_{\text{av}}^{\text{tel}} = \frac{2 - p}{2} \qquad (16)$$

from which it easily follows that in the limit of the maximally entangled pure state ($p = 1$) the channel is ideal, while for the maximally mixed state ($p = 0$), the fidelity falls below the classical limit of $2/3$.

In figure 1, we plot the average teleportation fidelity $W_{\text{av}}^{\text{tel}}$ against the channel parameter p. A similar plot is also provided for the average teleportation fidelity $F_{\text{av}}^{\text{tel}}$ corresponding to the non-orthogonal entangled channel with respect to the parameter r, which also ranges between 0 and 1. Note that in the region where both the fidelities exceed the classical limit of $2/3$, the non-orthogonal channel always outperforms the Werner state. Since the latter is an example of a maximally entangled mixed state, it is apparent that using a non-orthogonal channel leads to a better efficiency of teleportation, on average, compared to using a noisy channel.

Let us now consider an orthogonal non-maximally entangled state as the channel for the teleportation of a single
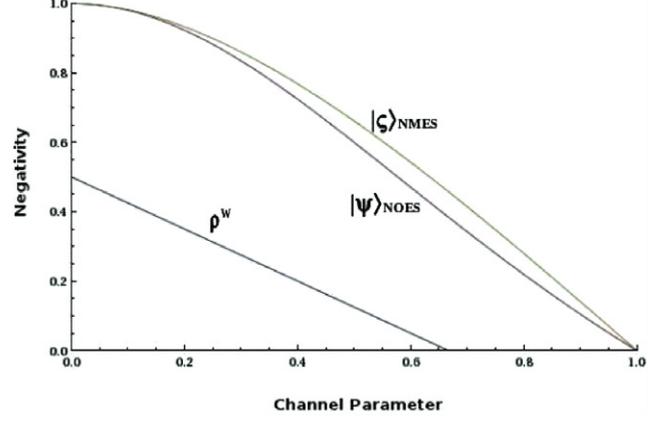

Figure 2. The magnitude of entanglement of the three channels, namely the non-orthogonal state $|\psi\rangle_{\text{NOES}}$, the (mixed) Werner state ρ^W and the non-maximally entangled pure state $|\xi\rangle_{\text{NMES}}$, is plotted against their respective channel parameters r, p and s for comparison. The state $|\xi\rangle_{\text{NMES}}$ is more entangled compared to $|\psi\rangle_{\text{NOES}}$, which in turn is more entangled than ρ^W.

qubit state (9). This non-maximally entangled state can be written as

$$|\zeta\rangle_{ab} = u|01\rangle + v|10\rangle, \qquad (17)$$

where $u^2 + v^2 = 1$. Here we note the following facts: (i) if $u = 0$, then the two-qubit state (17) becomes separable, (ii) if $u = \frac{1}{\sqrt{2}}$, then the two-qubit state reduces to a maximally entangled state and (iii) if $0 < u < \frac{1}{\sqrt{2}}$, then the state is a non-maximally entangled state. The parameter u is a measure of non-maximality. Our aim is to compare the efficiency of this channel with the non-orthogonal channel studied in section 2. In order to keep the two parameters u (for the non-maximally entangled channel) and r (for the non-orthogonal channel) on the same footing, we have to re-scale the parameter u in such a way that it can assume values between 0 and 1. The re-scaled parameter s can be written as

$$s = 1 - \sqrt{2}u. \qquad (18)$$

Hence, $s = 0$ and $s = 1$ denote the maximally entangled state and separable state, respectively. All other values of the parameter s lying between 0 and 1 correspond to non-maximally entangled states.

Now, repeating the conventional teleportation protocol for the orthogonal system using the non-maximally entangled state (17) as a teleportation channel, we can obtain the teleportation fidelity in terms of the parameter s as

$$G^{\text{tel}} = 1 - 2y^2(1 - y^2)(1 - (1 - s)\sqrt{2 - (1 - s)^2}). \quad (19)$$

Clearly, the teleportation fidelity G^{tel} is input state dependent. Hence, for some input state it gives a better fidelity than some other states. So, as in the previous case of the non-orthogonal channel, it would be better to consider the average fidelity. The average teleportation fidelity over all input states is given by

$$G_{\text{av}}^{\text{tel}} = \frac{2 + (1 - s)\sqrt{2 - (1 - s)^2}}{3}. \qquad (20)$$
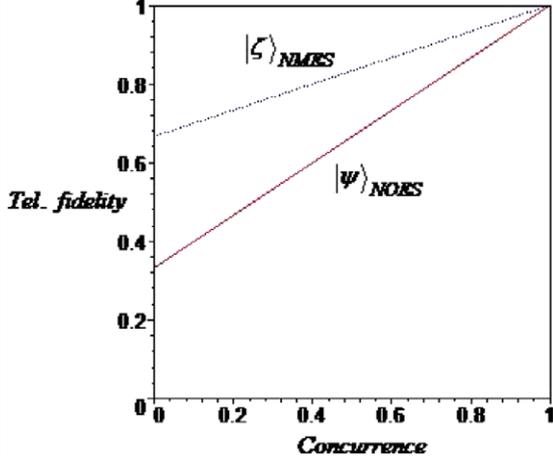
Figure 3. The average fidelity of teleportation of the the non-orthogonal channel $|\psi\rangle_{\text{NOES}}$, and the non-maximally entangled channel $|\xi\rangle_{\text{NMES}}$ is plotted versus the amount of entanglement in these states.

Note in figure 1 that this channel performs better teleportation compared to the non-orthogonal entangled state. Such a result can be understood by observing the magnitude of entanglement of these channels as functions of their respective parameters. The concurrence of all three channels is plotted in figure 2, which shows that the non-maximally entangled pure state is more entangled for a given parameter value compared to both the non-orthogonal and the mixed state. To complete the argument as to why the non-maximally entangled state is more efficient than the non-orthogonal state as a teleportation channel, in figure 3 we plot the average teleportation fidelity corresponding to $|\psi\rangle_{\text{NOES}}$ and $|\xi\rangle_{\text{NMES}}$ as a function of their respective concurrences. It is seen that for a given magnitude of entanglement the state $|\xi\rangle_{\text{NMES}}$ outperforms the state $|\psi\rangle_{\text{NOES}}$ as a teleportation channel.

4. Efficiency of teleportation by mixed orthogonal and non-orthogonal states

In the previous sections, we have investigated a pure non-orthogonal entangled state as a teleportation channel and compared its performance with that of a pure non-maximally entangled state as well as with a maximally entangled mixed state. A natural question that might arise then is how does a mixed non-orthogonal entangled channel fare in such a scheme of teleportation. To address this issue, let us now consider a mixed state involving non-orthogonal basis vectors, which can be defined as

$$\rho^N = g|\psi\rangle\langle\psi| + \frac{1-g}{4}I$$

$$= \begin{pmatrix} \frac{1-g}{4} + 4N_1^2r^2g & \frac{2gN_1^2\langle\alpha|\beta\rangle}{N_G} & \frac{2gN_1^2\langle\alpha|\beta\rangle}{N_G} & 0 \\ \frac{2gN_1^2\langle\alpha|\beta\rangle^*}{N_G} & \frac{1-g}{4} + g(\frac{N_1}{N_G})^2 & g(\frac{N_1}{N_G})^2 & 0 \\ \frac{2gN_1^2\langle\alpha|\beta\rangle^*}{N_G} & g(\frac{N_1}{N_G})^2 & \frac{1-g}{4} + g(\frac{N_1}{N_G})^2 & 0 \\ 0 & 0 & 0 & \frac{1-g}{4} \end{pmatrix},$$

$$0 \leqslant g \leqslant 1 \tag{21}$$

where $|\psi\rangle = \frac{N_1}{N_G}(|0\rangle|1\rangle + |1\rangle|0\rangle) + 2N_1\langle\alpha|\beta\rangle|0\rangle|0\rangle$, $N_1 = \frac{1}{\sqrt{2(1+r^2)}}$, $N_G = \frac{1}{\sqrt{1-r^2}}$, $\langle\alpha|\beta\rangle = re^{i\theta}$ and I is the identity operator.

For two-qubit systems, the negativity defined as twice the largest negative eigenvalue of the partially transposed density matrix may be used to quantify entanglement. The negativity of ρ^N is given by

$$\mathcal{N} = 2\max(0, -\lambda_N), \tag{22}$$

where λ_N denotes the negative eigenvalue of $(\rho^N)^{\text{PT}}$, PT denoting the partial transposition. Here we have

$$\mathcal{N} = \frac{g(3-r^2) - (1+r^2)}{2(1+r^2)} \quad \text{when} \quad \frac{1+r^2}{3-r^2} < g \leqslant 1. \tag{23}$$

For an entangled state ρ^N, the parameter g can be written as

$$g = \frac{1+r^2}{3-r^2} + \epsilon, \quad \epsilon > 0. \tag{24}$$

If $\epsilon < 0$, the state ρ^N is separable. Note that the condition $0 \leqslant g \leqslant 1$ does not hold when $\epsilon > \frac{2}{3}$.

Now, it is known [26] that any mixed spin-$\frac{1}{2}$ state ρ is useful for (standard) teleportation if and only if

$$\nu(\rho) = \sum_{i=1}^{3} \sqrt{u_i} > 1. \tag{25}$$

For the two-qubit state ρ^N, $\nu(\rho^N)$ is given by

$$\nu(\rho^N) = 1 + \frac{(3-r^2)\epsilon}{1+r^2}. \tag{26}$$

It follows that the two-qubit state described by the density matrix ρ^N can always be used as a teleportation channel when $0 < \epsilon < \frac{2}{3}$. The efficiency of the teleportation channel is measured by the average teleportation fidelity, which is given by

$$f_{\text{av}}^{\text{tel}}(\rho^N) = \frac{1}{2}\left[1 + \frac{\nu(\rho^N)}{3}\right]$$

$$= \frac{2}{3} + \frac{(3-r^2)\epsilon}{6(1+r^2)}. \tag{27}$$

For a given value of ϵ (or g), from the above expression one may compute the fidelity in terms of the non-orthogonality parameter r.

Next, we would like to compare the performance as a teleportation channel of the above mixed non-orthogonal entangled state with a mixed non-maximally entangled state. As an example of the latter, we choose a state proposed recently [25] given by a convex combination of a separable density matrix $\rho_{12}^G = \text{Tr}_3(|\text{GHZ}\rangle_{123})$ and an inseparable density matrix $\rho_{12}^W = \text{Tr}_3(|W\rangle_{123})$ as

$$\rho_{\text{new}} = p\rho_{12}^G + (1-p)\rho_{12}^W, \quad 0 \leqslant p \leqslant 1, \tag{28}$$

where $|\text{GHZ}\rangle$ and $|W\rangle$ denote the three-qubit GHZ state [27] and the W state [28], respectively. This construction is somewhat similar in spirit to the Werner state which
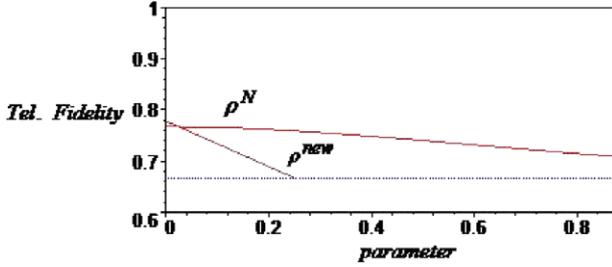
Figure 4. The average fidelity of teleportation for the mixed non-orthogonal channel $f_{av}^{tel}(\rho^N)$ (with $\epsilon = 0.2$) and for the non-maximally entangled mixed channel $f_{av}^{tel}(\rho_{new})$ is plotted against their respective channel parameters r and p. The horizontal dotted line corresponds to the classical teleportation fidelity of $2/3$.
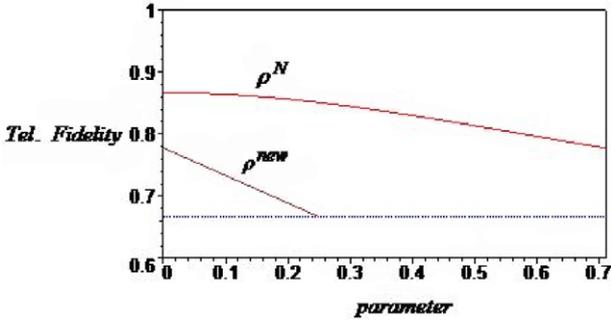

Figure 5. The average fidelity of teleportation for the the mixed non-orthogonal channel $f_{av}^{tel}(\rho^N)$ (with $\epsilon = 0.4$), and the non-maximally entangled mixed channel $f_{av}^{tel}(\rho_{new})$ is plotted versus their respective channel parameters r and p. The horizontal dotted line corresponds to the classical teleportation fidelity of $2/3$.

is a convex combination of a maximally mixed state and a maximally entangled pure state. Note that the GHZ state and the W state are two-qubit separable and inseparable states, respectively, when a qubit is lost from the corresponding three qubit states. The state ρ_{new} was studied as a teleportation channel in [25] where it was observed that it leads to a better teleportation efficiency compared to some other non-maximally entangled mixed states. The average teleportation fidelity corresponding to ρ_{new} is given by

$$f_{av}^{tel}(\rho_{new}) = \frac{7-4p}{9}, \quad 0 \leqslant p < \frac{1}{4} \qquad (29)$$

with $\frac{2}{3} < f_{av}^{tel}(\rho_{new}) \leqslant \frac{7}{9}$. The state ρ_{new} cannot be used as a efficient teleportation channel for $p \geqslant 1/4$, since in this case the teleportation fidelity falls below the classical fidelity.

Finally, we present a comparison of the average teleportation fidelities between the non-orthogonal mixed channel ρ^N and the non-maximally entangled mixed channel ρ_{new}. Their fidelities are plotted against the respective channel parameters r and p for two different values of the parameter ϵ in figures 4 and 5. Note that the non-orthogonal mixed channel could lead to a better output compared to the non-maximally entangled mixed channel for a range of parameter values. This feature is different from the case of pure states studied in the earlier section, where we saw that the non-maximally entangled pure channel is always more efficient than the non-orthogonal pure channel.

5. Conclusions

The aim of this work was to investigate the use of non-orthogonal entangled states as the resource for performing the teleportation of an unknown qubit. To this end, we first extended the standard teleportation protocol [2] to the case of non-orthogonal basis states. Since the non-orthogonal channel is less entangled than the corresponding orthogonal case, teleportation through it leads to a loss of fidelity. We obtained the expression for the average teleportation fidelity corresponding to a non-orthogonal channel that is independent of the input states. Subsequently, we made a comparison of the teleportation efficiency of the non-orthogonal channel with other non-ideal channels corresponding to mixed as well as non-maximally entangled states. We have also presented a comparative study of the teleportation fidelity of mixed states.

For making the above comparisons, we have defined suitable parameters in order to bring out the quantitative features of this study. Of course, the parameters p (mixedness parameter), r (nonorthogonality parameter) and s (non-maximality parameter) have different meanings, but the sense in which we have made the comparison can be summarized as follows. While the parameters r and p vary between 0 and 1, the parameter s has been re-scaled to restrict its value from 0 to 1. Then, for any given value of such a parameter within $[0, 1]$, the teleportation fidelity and the magnitude of entanglement are compared for different cases. It is on the basis of such a comparison that the statement about the relative efficiency of teleportation fidelity for different cases was made in the paper. A possible application of these results could be in evaluating the performance of teleportation channels where more than one of the above three different types of departure from ideal settings (maximally entangled pure orthogonal states) is present.

Our analysis has yielded many interesting results. First we find that a non-orthogonal entangled channel could be more efficient as a teleportation resource compared to a noisy channel. For the latter, we consider the example of the Werner channel which is a maximally entangled mixed state. This result follows from the fact that for a given range of channel parameter values, the Werner state, in spite of being a maximally entangled mixed state, is nevertheless less entangled than the corresponding non-orthogonal state. Our next comparative study pertaining to a non-maximally entangled pure state shows that the latter is always more efficient in performing teleportation compared to the non-orthogonal entangled state. We finally presented a comparison of mixed states in the orthogonal and the non-orthogonal basis. Here we compared the average teleportation fidelity of a non-orthogonal mixed state with that of a non-maximally entangled mixed state [25]. Contrary to the case of pure states, we find here that a suitable choice of parameter values could lead to the non-orthogonal channel performing better teleportation than the non-maximally entangled mixed state. It would be interesting to extend our study to the case of higher dimensions and multipartite states. Such investigations could be useful in devising channels for practical teleportation where it is almost impossible to work with ideal channels.

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